II PORTUGUESE STATA USER GROUP MEETING
SEPTEMBER 7TH, 2012
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Cox Models in Risk Management

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AGENDA

- Mathematical functions in survival analysis
- Cox proportional hazard model
- Optimal Age Replacement
- Risk Management (generalisation)
- Cox model in risk management
- Weibull probability function
- Examples
- Conclusions

Mathematical functions in survival analysis

Survival function, *S*(t)

$$S(t) = \Pr(T > t)$$

T is survival time, a random variable of the time to event since the origin (t=0)

Cumulative distribution function, *F*(t)

$$F(t) = 1 - S(t)$$

Hazard function, *h*(t)

$$h(t) = \lim_{\delta t \to 0} \frac{\Pr(t < T < t + \delta t \mid T \geq t)}{\delta t} = -\frac{d \log S(t)}{dt}$$

Cumulative hazard function, *H*(t)

$$H(t) = \int_0^t h(u) du$$

$$H(t) = -\ln\{S(t)\}$$

$$S(t) = e^{-H(t)}$$

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Cox proportional hazard model

Hazard function for the *i*th individual,

$$h_i(t \mid x_i) = h_0(t)\exp(x_i\beta)$$

where $x_i = x_{i1}, \ldots, x_{ik}$ **is the covariates vector**

$\beta = \beta_1, \ldots, \beta_k$ **is the vector of regression coefficients.**

$h_0(t) = h_i(t \mid x = 0)$ **is the baseline hazard function.**

The Hazard Rate of the associated to the j covariate,

$$HR_j = \exp(x_{ij}\beta_j)$$

The covariates can be time dependent.

Optimal Age Replacement

In maintenance management the optimal time t_{op} for taking a preventive operation that minimizes the system average operational cost Φ_{OC} in the long run [1] corresponds to the minimum of:

$$\Phi_{OC}(t) = \frac{C_P S(t) + C_F F(t)}{\int_0^t S(\tau) d\tau}$$

C_P **- Preventive repair cost:** average cost per item due to repairing a defect prior to failure occurrence, including all materials and labor costs.

C_F ***- Failure repair costs:*** average cost of in service failure occurrence. Includes the cost of completely repairing the failed item plus all costs with materials, labor, loss of production, loss of image, etc., occurring with the system shutdown.

The time can be replaced for any other counter.

Risk Management (generalization)

All critical events that:

- hazard function is time dependent (increasing);
- there is a much less expensive preventive action that restores the initial hazard level.

<u>Is possible to calculate the optimal time to do it (the ideal length of the hazard cycle).</u>

This time length depends on the difference between the preventive action and the event occurrence costs and on the hazard function.

Cox model in risk management

The classical single variable hazard functions can the generalized to a multivariate probability model.

Single variables: lead to one optimal hazard cycle length.

Cox models with fixed covariates: move the event optimal cycle length and cost values.

Cox models with time dependent covariates: the optimal cycle length and cost are successively change over time.

The ideal length of the hazard cycle, correspond to the time that minimized the expected cost per time.

$$\Phi_{i\mathrm{OP}}(t) = \frac{C_{\mathrm{P}} S_i\left(t \mid x_i(t)\right) + C_{\mathrm{F}} F_i\left(t \mid x_i(t)\right)}{\int_0^t S_i(\tau \mid x_i(\tau))\, d\tau}$$

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Weibull probability function

This methodology allows the use of any parametric or non parametric probability, for baseline hazard function.

In the examples below the Weibull probability function is used, because of its simplicity and allowance to easy control the increasing degree (or the decreasing) of the occurrence rates:

$$h_0(t) = \lambda \gamma t^{\gamma - 1}$$

where

λ **- scale parameter**

γ **- shape parameter.**

$\gamma < 1 \rightarrow$ decreasing failure rate
$\gamma = 1 \rightarrow$ constant failure rate
$\gamma > 1 \rightarrow$ increasing failure rate

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Examples

For simplification all the costs and times are nondimensionalized, making the preventive action cost, $C_P=1$ *monetary unit* and the scale parameter of the Weibull model $\lambda=1$.

Starting by using the Cox models with fixed covariates, it can be measured the impact of each covariate on the event optimal cycle length and cost values.

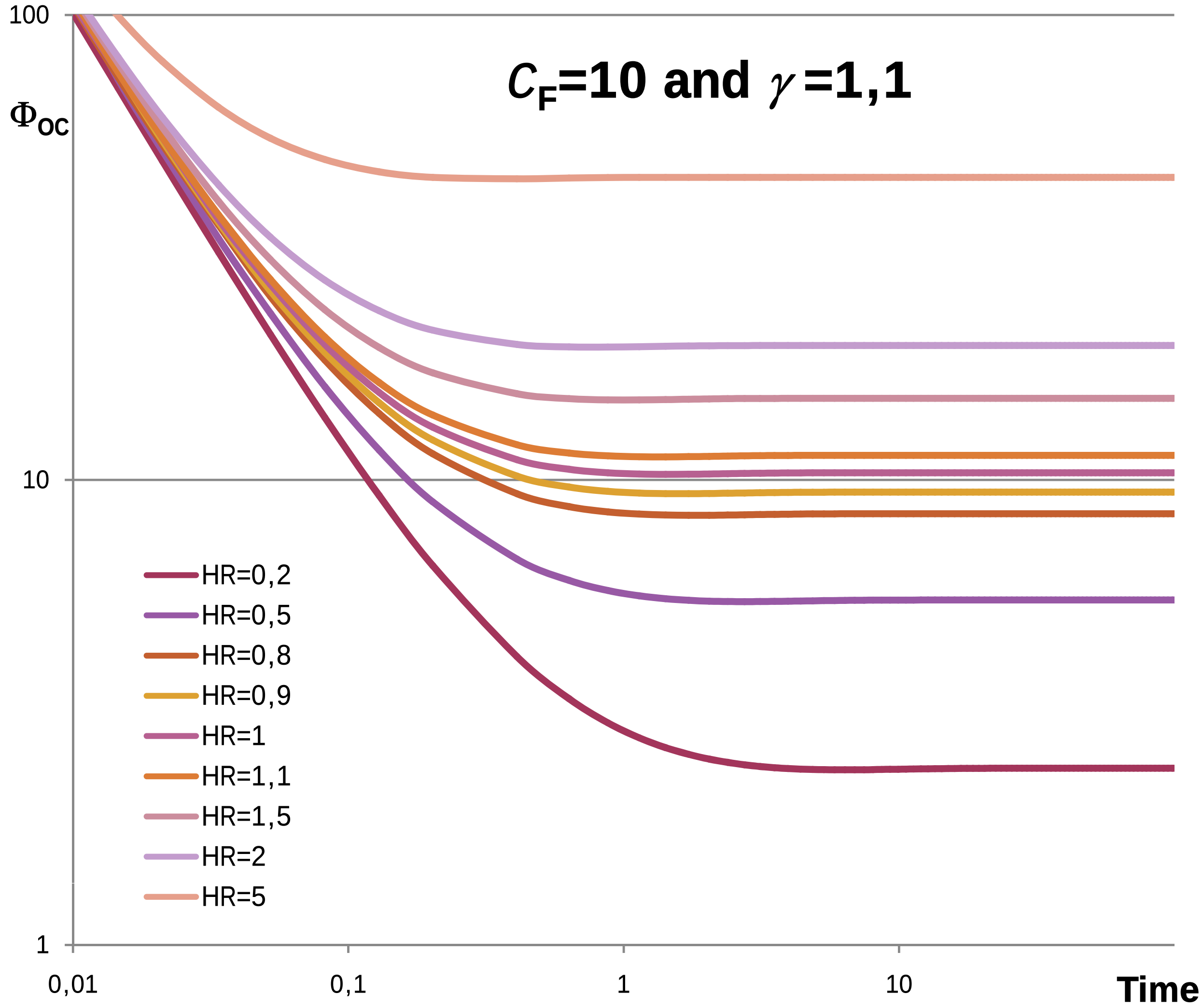
In the next 3 tables and graphics are present the results for a set Hazard Rates, for 3 different baseline hazard functions $\{\gamma=1.1, \gamma=2, \gamma=5\}$ and 3 event occurrence costs, $\{C_F=3, C_F=5, C_F=10\}$.

Fixed Hazard Rates (γ=1,1)

γ=1.1	C_F=3		C_F=5		C_F=10	
HR	t_{op}	$Min\ \Phi(t)$	t_{op}	$Min\ \Phi(t)$	t_{op}	$Min\ \Phi(t)$
0.2	100,7	0,72	22,1	1,20	6,3	2,38
0.5	44,5	1,65	9,6	2,76	2,7	5,48
0.8	28,7	2,54	6,3	4,23	1,8	8,39
0.9	25,7	2,82	5,6	4,71	1,6	9,34
1	23,8	3,11	5,1	5,18	1,5	10,28
1.1	21,7	3,39	4,7	5,65	1,3	11,21
1.5	16,3	4,49	3,5	7,49	1,0	14,87
2	12,4	5,84	2,7	9,73	0,8	19,31
5	5,5	13,42	1,2	22,38	0,3	44,41

Example with γ=1,1 and C_F=10

HR	t_{op}	$\Phi(t_{OP})$
0.2	6,3	2,38
0.5	2,7	5,48
0.8	1,8	8,39
0.9	1,6	9,34
1	1,5	10,28
1.1	1,3	11,21
1.5	1,0	14,87
2	0,8	19,31
5	0,3	44,41



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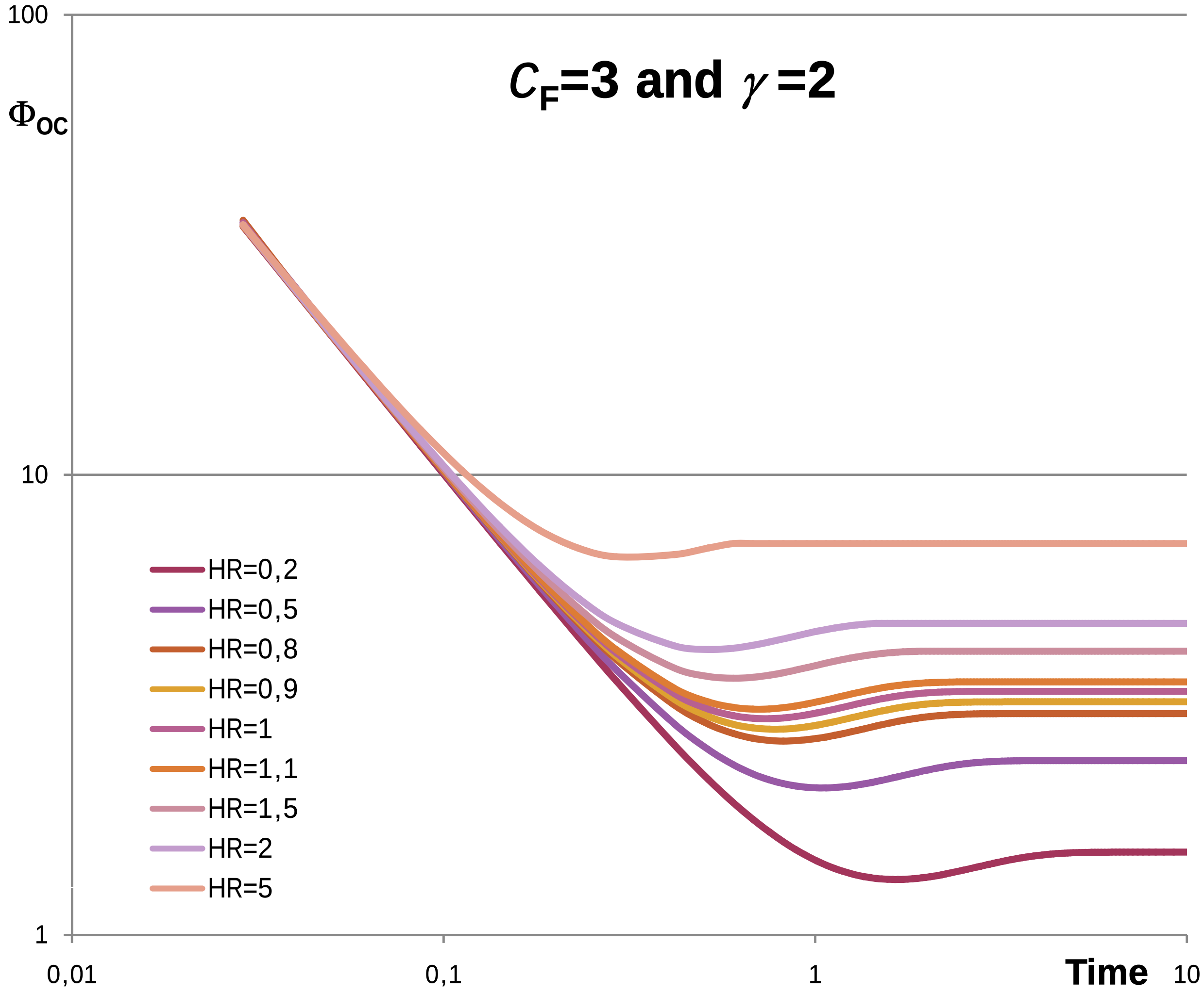
Fixed Hazard Rates (γ=3)

γ=3	C_F=3		C_F=5		C_F=10	
HR	t_{op}	$Min\ \Phi(t)$	t_{op}	$Min\ \Phi(t)$	t_{op}	$Min\ \Phi(t)$
0.2	1,65	1,32	1,14	1,83	0,75	2,71
0.5	1,04	2,09	0,72	2,89	0,48	4,28
0.8	0,83	2,64	0,57	3,65	0,38	5,42
0.9	0,78	2,80	0,54	3,88	0,35	5,75
1	0,74	2,95	0,51	4,09	0,34	6,06
1.1	0,70	3,10	0,49	4,28	0,32	6,35
1.5	0,60	3,62	0,42	5,00	0,27	7,42
2	0,52	4,17	0,36	5,78	0,24	8,56
5	0,33	6,60	0,23	9,13	0,15	13,54

Example with γ=2 and C_F=3

HR	t_{op}	$\Phi(t_{OP})$
0.2	1,65	1,32
0.5	1,04	2,09
0.8	0,83	2,64
0.9	0,78	2,80
1	0,74	2,95
1.1	0,70	3,10
1.5	0,60	3,62
2	0,52	4,17
5	0,33	6,60



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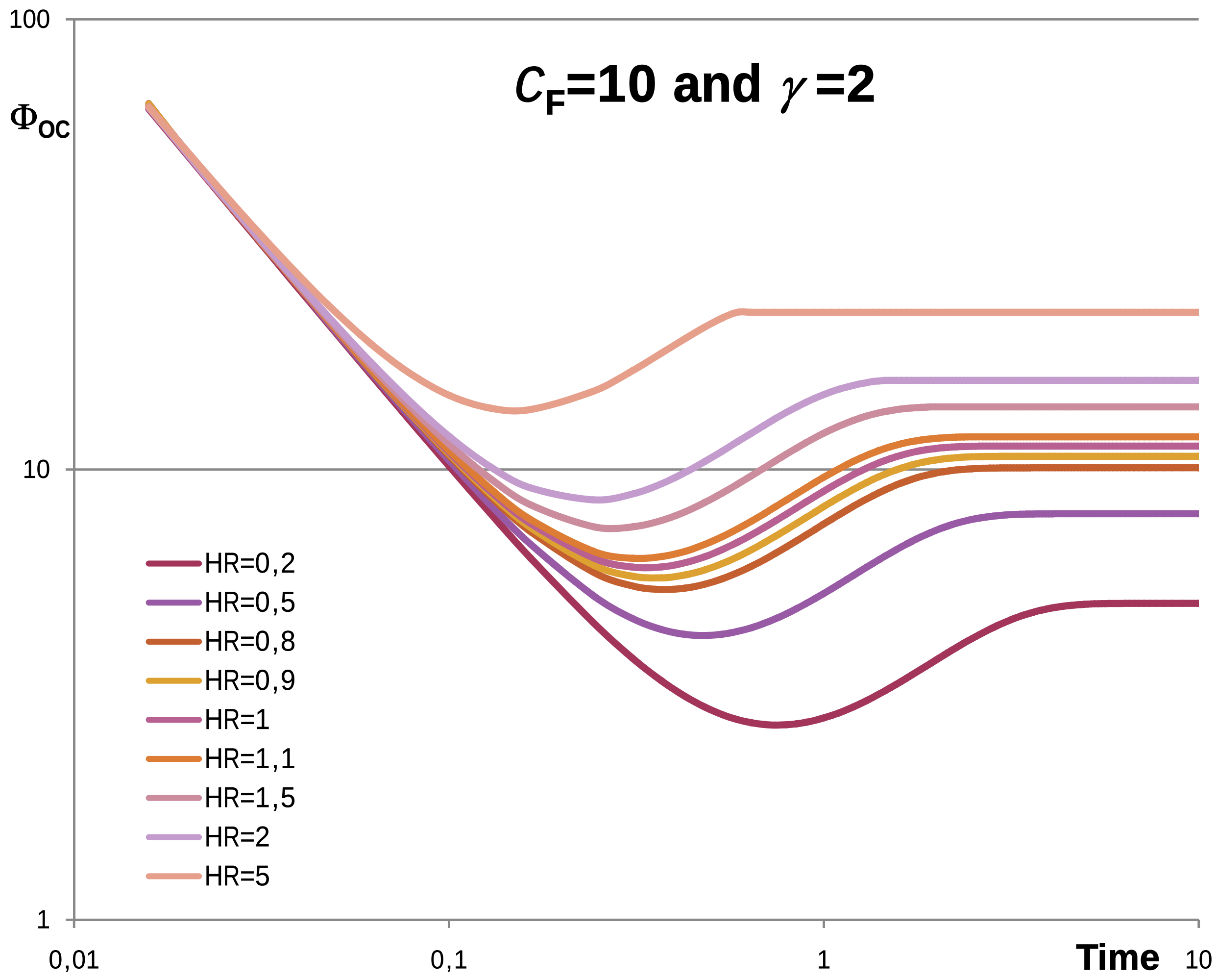
Example with γ=2 and C_F=10

HR	t_{op}	$\Phi(t_{OP})$
0.2	0,75	2,71
0.5	0,48	4,28
0.8	0,38	5,42
0.9	0,35	5,75
1	0,34	6,06
1.1	0,32	6,35
1.5	0,27	7,42
2	0,24	8,56
5	0,15	13,54



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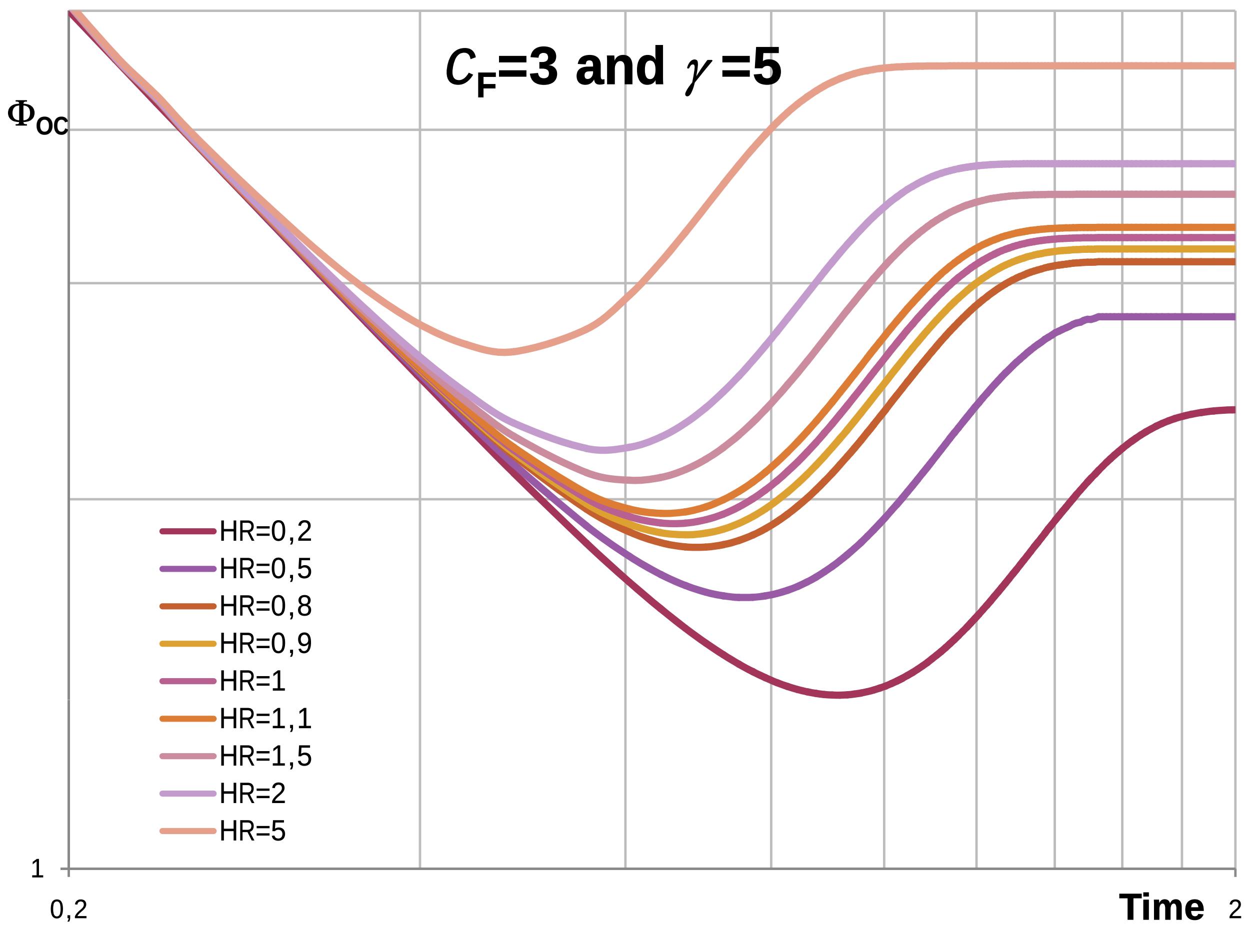
Fixed Hazard Rates (γ=5)

$\gamma = 5$	C_F=3		C_F=5		C_F=10	
HR	t_{op}	$Min\ \Phi(t)$	t_{op}	$Min\ \Phi(t)$	t_{op}	$Min\ \Phi(t)$
0.2	0,91	1,38	0,79	1,58	0,67	1,86
0.5	0,76	1,66	0,66	1,90	0,56	2,23
0.8	0,69	1,83	0,60	2,09	0,51	2,45
0.9	0,68	1,87	0,59	2,14	0,50	2,51
1	0,66	1,91	0,57	2,19	0,49	2,56
1.1	0,65	1,95	0,56	2,23	0,48	2,61
1.5	0,61	2,07	0,53	2,37	0,45	2,78
2	0,58	2,19	0,50	2,51	0,43	2,95
5	0,48	2,64	0,42	3,02	0,35	3,54

Example with γ=5 and C_F=3

HR	t_{op}	$\Phi(t_{OP})$
0.2	0,91	1,38
0.5	0,76	1,66
0.8	0,69	1,83
0.9	0,68	1,87
1	0,66	1,91
1.1	0,65	1,95
1.5	0,61	2,07
2	0,58	2,19
5	0,48	2,64



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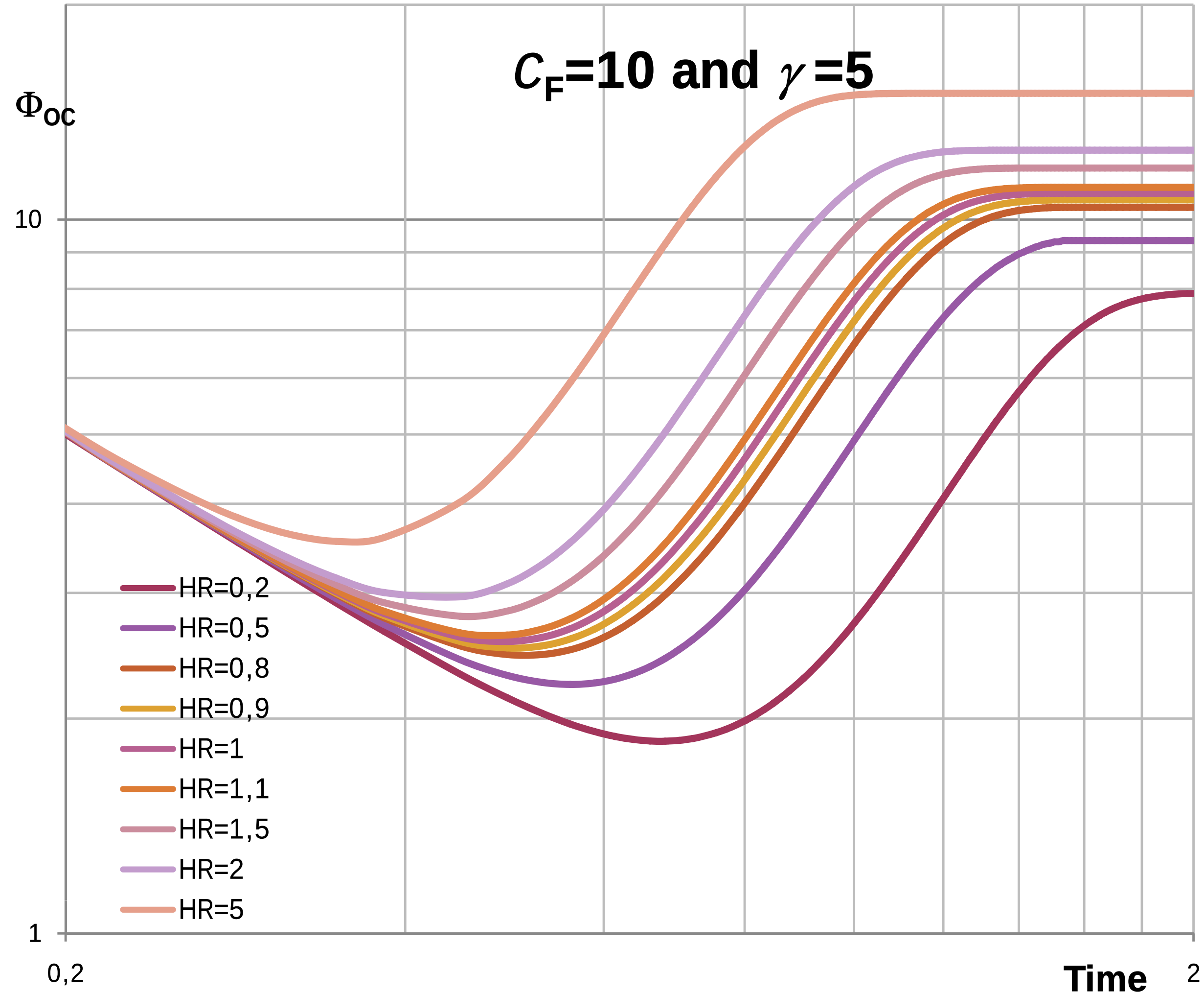
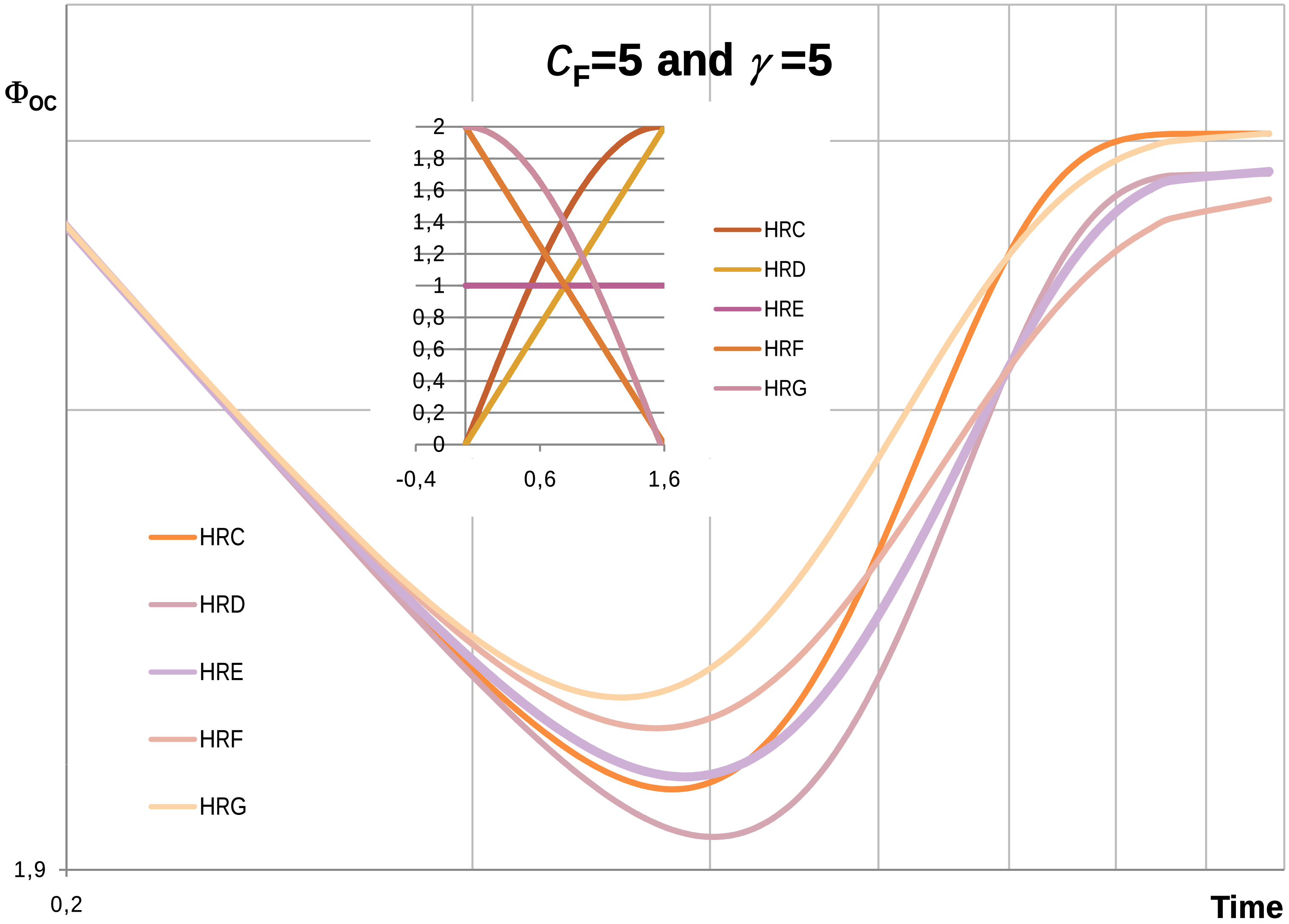
Example with γ=5 and C_F=10

HR	t_{op}	$\Phi(t_{OP})$
0.2	0,67	1,86
0.5	0,56	2,23
0.8	0,51	2,45
0.9	0,50	2,51
1	0,49	2,56
1.1	0,48	2,61
1.5	0,45	2,78
2	0,43	2,95
5	0,35	3,54



Time dependent HRs (γ=5 and C_F=5)



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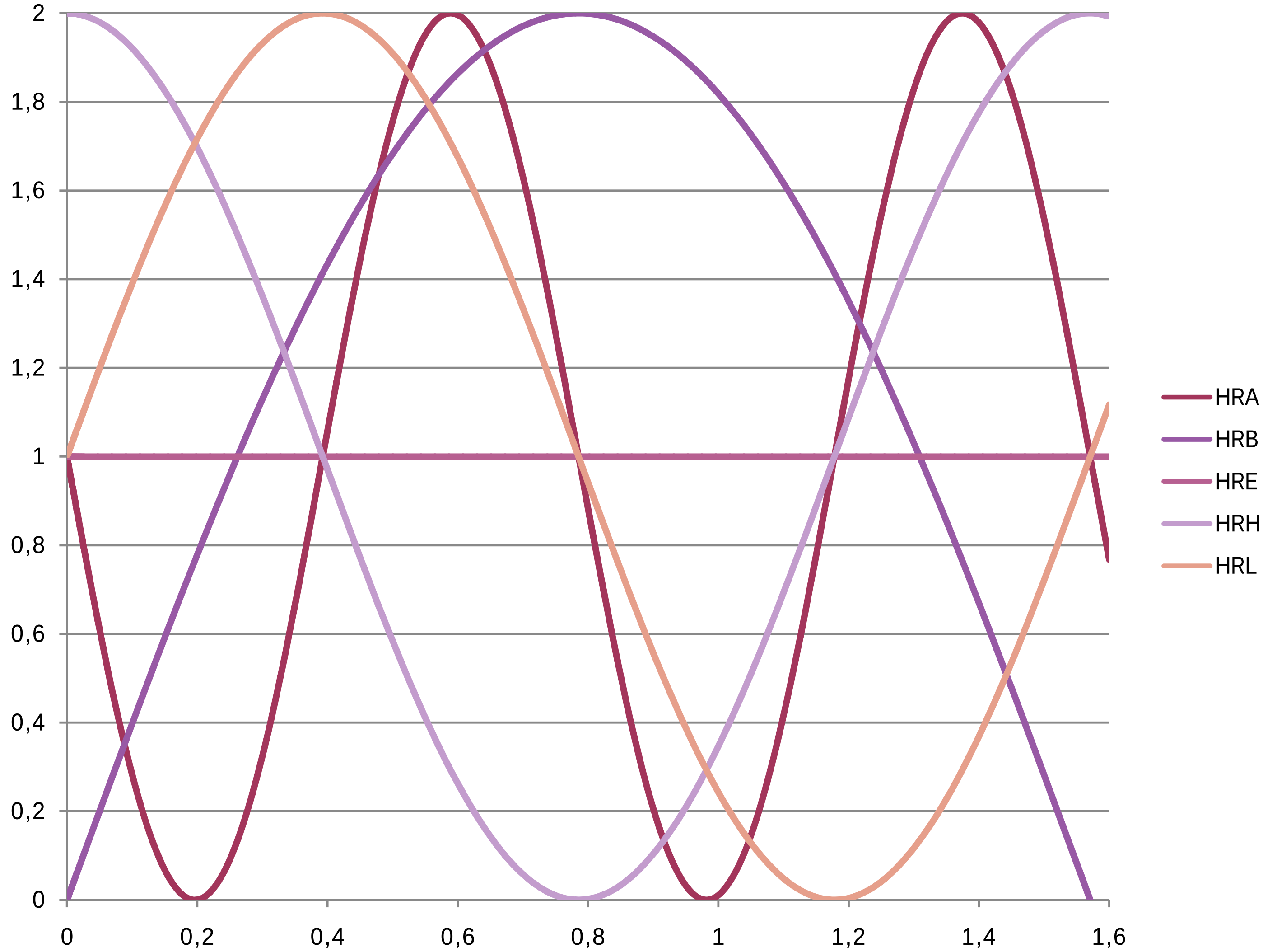
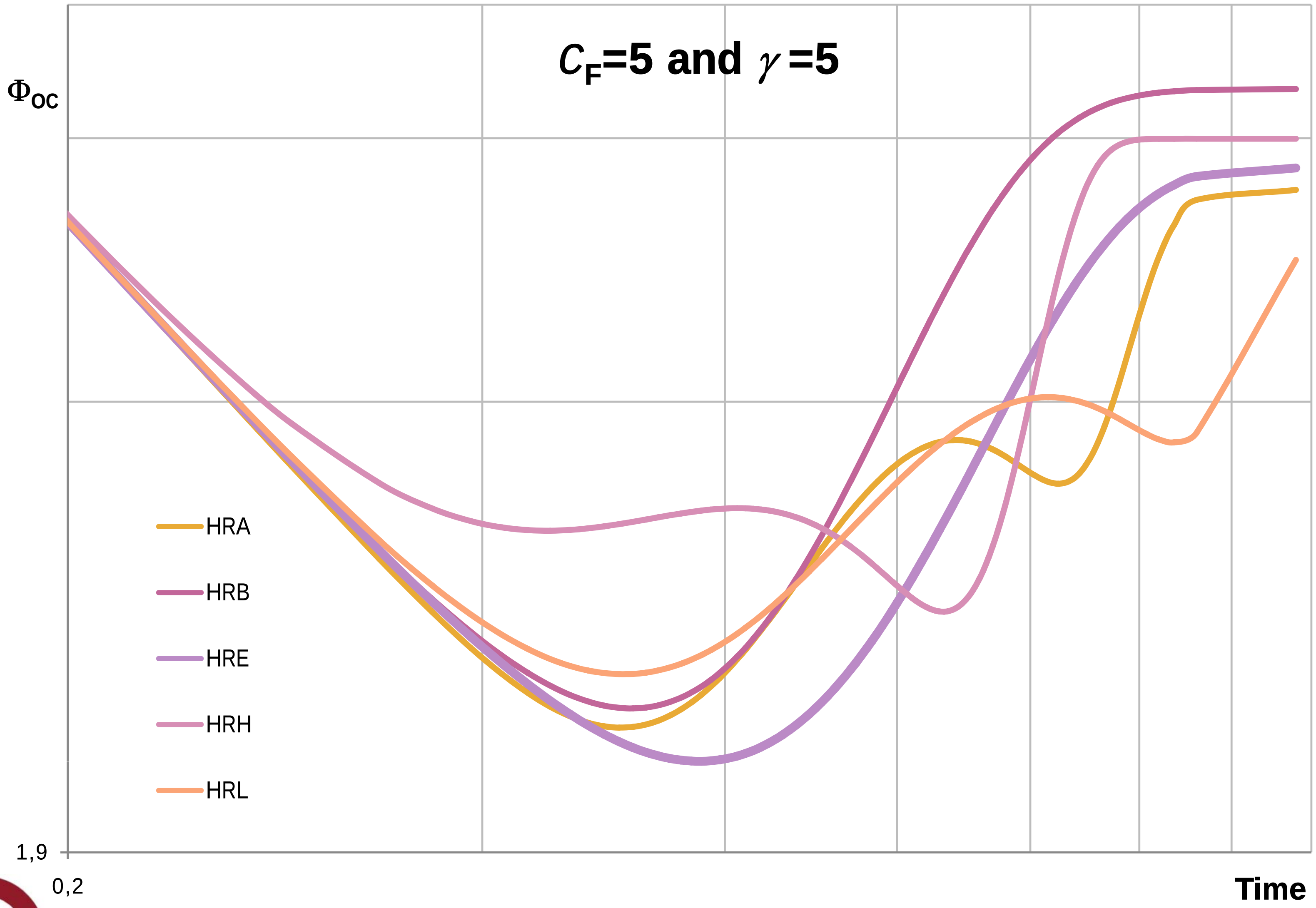
Other Hazard Rate time functions



Time dependent HRs (γ=5 and C_F=5)



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Conclusions

All critical events that the hazard increases with time and there is a preventive action that restores the initial hazard level. <u>Is possible to calculate the optimal time to do it (the ideal length of the hazard cycle).</u>

This time length depends on the difference between the preventive action and the event occurrence costs and on the hazard function.

<u>The Cox probability model can be used as the hazard functions. There are two variants:</u>

- **Fixed covariates:** move the event optimal cycle length and cost values. (HR<1→Delay, HR>1→Anticipate)
- **Time dependent covariates:** the optimal cycle length and cost are successively change over time.

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Thanks for your attention

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