The Hierarchy of Factor Invariance

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2013 Stata Conference New Orleans

July 19, 2013

Apologia

This presentation is in html because it was too difficult to fit all of the CFA output onto traditional slides. And, yes, there is probably too much output shown.

Introduction

Whenever researchers conduct confirmatory factor analysis with multiple groups, the issue of factor or measurement invariance comes up.

At its most basic level, factor invariance is whether the factors in each group are measuring the same thing. There are a number of ways in which invariance can be assessed. The table below presents the hierarchy of factor invariance in ordered by the number of constraints placed on the model, running from the fewest to the most constraints. The constraints are cumulative through the hierarchy.

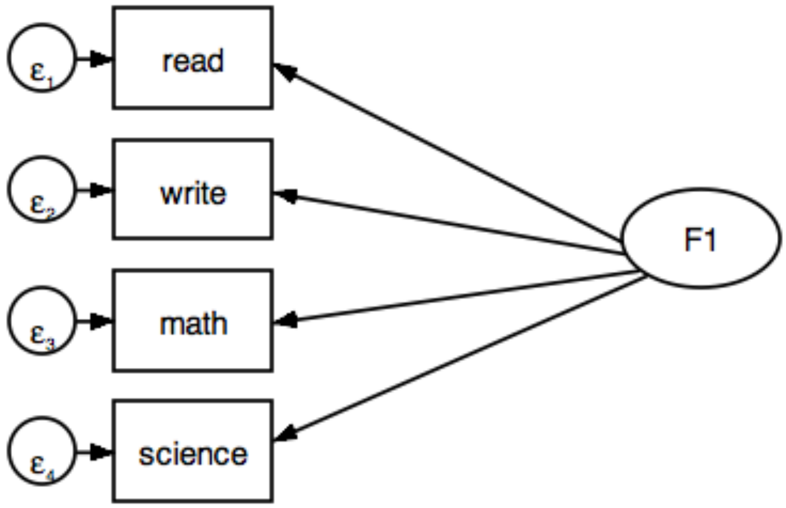
The Hierarchy of Factor Invariance

```
1  Configurational(Dimensional) Invariance -- equal number of factors
2  Metric (Pattern) Invariance             -- equal loadings
3  Strong (Scalar) Invariance              -- equal intercepts
4  Strict Invariance                       -- equal residuals
5  Strict Invariance + Equal Factor Means
6  Strict Invariance + Equal Factor Means & Factor Variances
```

The Data

```
use http://www.ats.ucla.edu/stat/data/hsbdemo,clear
```

The dataset contains four indicator variables (read, write, math & science) that we will use in our single-factor model. The diagram for a single-group model looks like this:



We will use female (0=male, 1=female) as the multiple-group indicator.

Configurational (Dimensional) Invariance

Configurational or dimensional factor invariance implies that the number of factors are the same within each group. One way we can verify this is by running separate PCAs for each group.

```
. pca read write math science if female==0
. pca read write math science if female==1
```

```
[Edited Output}
    ---------------------------------------------------------------------------
                 |   Male               |   Female
       Component |   Eigenvalues        |   Eigenvalues
    -------------+----------------------+--------------------------------------
           Comp1 |      2.88982         |    2.95214
           Comp2 |       .409958        |     .421286
           Comp3 |       .375182        |     .3516
           Comp4 |       .325036        |     .274972
```

From the eigenvalues it seems rather clear that there is only one factor within each group. This is not a big surprise since each of the indicators are normed test scores in each of four academic areas.

Separate CFAs for each group

Before continuing on with the hierarchy of factor invariance, it is a good idea to run a CFA separately for each group to see if there are any problems estimating the models. The results for this step will be the same as the multiple-group analysis with all parameters allowed to vary freely and, as such, serves as a check that the multiple-group analysis is coded correctly.

For identification purposes **sem** will fix the loading of the first indicator to one. Additionally, we will also set the intercept and the residual variance for the first indicator to zero in to assist in estimating (identifying) the factor mean.

```
. sem (F1 -> read write math science)       ///
```

```
      (read <- F1@1 _cons@0) if female==0,  ///
      mean(F1) variance(e.read@0)
. sem (F1 -> read write math science)       ///
      (read <- F1@1 _cons@0) if female==1,  ///
      mean(F1) variance(e.read@0)

[Edited Output]

-------------------------------------------------------------------------------
             |        Males                 |       Females
             |                 OIM          |                 OIM
             |      Coef.   Std. Err.       |      Coef.   Std. Err.
-------------+------------------------------+----------------------------------
Measurement  |                              |
  read <-    |                              |
          F1 |          1  (constrained)    |          1  (constrained)
       _cons |          0  (constrained)    |          0  (constrained)
  -----------+------------------------------+----------------------------------
  write <-   |                              |
          F1 |   .6360156   .0782702        |    .5021136   .0607196
       _cons |   16.52388   4.214667        |   29.01451    3.199544
  -----------+------------------------------+----------------------------------
  math <-    |                              |
          F1 |   .5597399   .0765211        |    .64698     .0612728
       _cons |   23.37725    4.12048        |   18.92367   3.228695
  -----------+------------------------------+----------------------------------
  science <- |                              |
          F1 |   .6799764   .0799045        |    .5342854   .0692102
       _cons |   17.31158    4.30267        |   23.05656    3.646948
-------------+------------------------------+----------------------------------
Mean         |                              |
          F1 |   52.82418   1.095334        |   51.73394   .9589364
-------------+------------------------------+----------------------------------
Variance     |                              |
      e.read |          0  (constrained)    |          0   (constrained)
     e.write |   60.86516   9.023252        |    40.28008   5.456223
      e.math |   58.17521   8.624466        |    41.01741   5.556098
   e.science |   63.43344   9.403998        |    52.33273   7.088839
          F1 |  109.1779   16.18561         |   100.2319   13.57712
-------------+------------------------------+----------------------------------
              LR chi2(3)   =   31.05        |                  53.48
```

In general the factor loadings look fairly similar. However, there seems to be variability in some of the intercepts and residual variances.

Model 1: Free All Parameters

We begin by running a two-group model in which all of the parameters are allowed to vary freely. Note the **ginvariant(none)** option. No multiple-group model can fit better than this one. Thus, this model is used to see the effect of constraining the loadings for the metric invariance model.

```
. sem (F1 -> read write math science)   ///
      (F1 -> read@1)                    ///  /* set loadings  to 1 in both groups */
      (read <- F1 _cons@0),             ///  /* set intercept to 0 in both groups */
      group(female) ginvariant(none)    ///
      mean(F1) variance(e.read@0)       /* set residual var to zero in both groups */

Endogenous variables
Measurement:  read write math science
```

```
Exogenous variables
Latent:       F1
Fitting target model:

Iteration 0:   log likelihood = -25716.222
Iteration 11:  log likelihood = -2779.9729

Structural equation model                       Number of obs      =       200
Grouping variable  = female                     Number of groups   =         2
Estimation method  = ml
Log likelihood     = -2779.9729

 ( 1)  [read]0bn.female#c.F1 = 1
 ( 2)  [var(e.read)]0bn.female = 0
 ( 3)  [read]0bn.female = 0
 ( 4)  [read]1.female#c.F1 = 1
 ( 5)  [var(e.read)]1.female = 0
 ( 6)  [read]1.female = 0
------------------------------------------------------------------------------
             |                 OIM
             |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
Measurement  |
  read <-    |
    F1       |
         [*] |          1  (constrained)
    _cons    |
         [*] |          0  (constrained)
  -----------+----------------------------------------------------------------
  write <-   |
    F1       |
        male |   .6360156   .0782702     8.13   0.000     .4826089    .7894224
      female |   .5021136   .0607196     8.27   0.000     .3831054    .6211218
    _cons    |
        male |   16.52388   4.214667     3.92   0.000     8.263282    24.78447
      female |   29.01451   3.199544     9.07   0.000     22.74352     35.2855
  -----------+----------------------------------------------------------------
  math <-    |
    F1       |
        male |   .5597399   .0765211     7.31   0.000     .4097614    .7097185
      female |     .64698   .0612728    10.56   0.000     .5268875    .7670724
    _cons    |
        male |   23.37725    4.12048     5.67   0.000     15.30126    31.45325
      female |   18.92367   3.228694     5.86   0.000     12.59555    25.25179
  -----------+----------------------------------------------------------------
  science <- |
    F1       |
        male |   .6799764   .0799045     8.51   0.000     .5233665    .8365863
      female |   .5342854   .0692102     7.72   0.000     .3986359     .669935
    _cons    |
        male |   17.31158    4.30267     4.02   0.000       8.8785    25.74466
      female |   23.05656   3.646948     6.32   0.000     15.90867    30.20444
-------------+----------------------------------------------------------------
Mean         |
  F1         |
        male |   52.82418   1.095334    48.23   0.000     50.67736    54.97099
      female |   51.73394   .9589366    53.95   0.000     49.85446    53.61343
-------------+----------------------------------------------------------------
Variance     |
  e.read     |
         [*] |          0  (constrained)
```

```
  e.write    |
        male |   60.86516   9.023252                      45.51748    81.38779
      female |   40.28008   5.456223                      30.88795     52.5281
  e.math     |
        male |   58.17521   8.624466                      43.50583    77.79084
      female |   41.01738   5.556092                      31.45334    53.48958
  e.science  |
        male |   63.43344   9.403998                      47.43815    84.82205
      female |   52.33273   7.088839                      40.13027     68.2456
  F1         |
        male |   109.1779   16.18561                      81.64773    145.9907
      female |    100.232   13.57713                      76.86081    130.7096
------------------------------------------------------------------------------
Note: [*] identifies parameter estimates constrained to be equal across
      groups.

LR test of model vs. saturated: chi2(6)   =     84.53, Prob > chi2 = 0.0000
```

Once again, you can visually compare the loadings, intercepts and residual variances between males and females.

Model 2: Metric Invariance (loadings invariant)

We check the metric or pattern invariance by constraining the loadings (path coefficients) to be equal across groups. We use the **ginvariant(mcoef)** option to do this. In the code below, **mcoef** stands for the measurement coefficient, i.e., the factor loading.

```
. sem (F1 -> read write math science)   ///
      (F1 -> read@1)                    ///  /* set loading  to 1 in both groups  */
      (read <- F1 _cons@0),             ///  /* set intercept to 0 in both groups */
      group(female)                     ///  /* multiple-group analysis */
      ginvariant(mcoef)                 ///  /* hold loadings equal */
      mean(F1)                          ///  /* estimate factor means */
      variance(e.read@0)                     /* fixed residual at 0 */

Endogenous variables
Measurement:  read write math science
Exogenous variables
Latent:       F1
Fitting target model:

Iteration 0:   log likelihood = -25597.924
Iteration 10:  log likelihood = -2782.2219

Structural equation model                        Number of obs      =       200
Grouping variable  = female                      Number of groups   =         2
Estimation method  = ml
Log likelihood     = -2782.2219

 ( 1)  [read]0bn.female#c.F1 = 1
 ( 2)  [write]0bn.female#c.F1 - [write]1.female#c.F1 = 0
 ( 3)  [math]0bn.female#c.F1 - [math]1.female#c.F1 = 0
 ( 4)  [science]0bn.female#c.F1 - [science]1.female#c.F1 = 0
 ( 5)  [var(e.read)]0bn.female = 0
 ( 6)  [read]0bn.female = 0
 ( 7)  [read]1.female#c.F1 = 1
 ( 8)  [var(e.read)]1.female = 0
 ( 9)  [read]1.female = 0
------------------------------------------------------------------------------
```

```
             |                 OIM
             |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
Measurement  |
  read <-    |
    F1       |
         [*] |          1  (constrained)
    _cons    |
         [*] |          0  (constrained)
  -----------+----------------------------------------------------------------
  write <-   |
    F1       |
         [*] |   .5522246   .0485991    11.36   0.000     .4569722    .6474769
    _cons    |
        male |   20.95007   2.695888     7.77   0.000     15.66623    26.23392
      female |   26.42207   2.587114    10.21   0.000     21.35142    31.49272
  -----------+----------------------------------------------------------------
  math <-    |
    F1       |
         [*] |   .6129485    .048102    12.74   0.000     .5186703    .7072267
    _cons    |
        male |   20.56656   2.664416     7.72   0.000      15.3444    25.78871
      female |   20.68425   2.563209     8.07   0.000     15.66045    25.70805
  -----------+----------------------------------------------------------------
  science <- |
    F1       |
         [*] |   .5965753   .0530537    11.24   0.000     .4925918    .7005587
    _cons    |
        male |   21.71717   2.925668     7.42   0.000     15.98297    27.45138
      female |   19.83406   2.831422     7.00   0.000     14.28457    25.38354
-------------+----------------------------------------------------------------
Mean         |
  F1         |
        male |   52.82417   1.095333    48.23   0.000     50.67736    54.97099
      female |   51.73396   .9589334    53.95   0.000     49.85449    53.61344
-------------+----------------------------------------------------------------
Variance     |
  e.read     |
         [*] |          0  (constrained)
  e.write    |
        male |   61.63169   9.180054                       46.0275    82.52599
      female |   40.53178   5.511979                      31.04841    52.91172
  e.math     |
        male |   58.48431   8.688283                      43.71062    78.25133
      female |   41.13349   5.581478                      31.52786    53.66568
  e.science  |
        male |   64.19285     9.5655                      47.93442    85.96582
      female |    52.7216   7.172172                      40.38242    68.83113
  F1         |
        male |   109.1777   16.18554                      81.64762    145.9903
      female |   100.2313   13.57695                      76.86044    130.7085
------------------------------------------------------------------------------
Note: [*] identifies parameter estimates constrained to be equal across
      groups.

LR test of model vs. saturated: chi2(9)   =     89.02, Prob > chi2 = 0.0000
```

We won't compare compare the chi-square and degrees of freedom to the previous model just yet. We will wait until after we run the models for the other forms of factor invariance.

Model 3: Strong Invariance (metric invariance plus equal intercepts)

As we step through each of the levels of invariance, we retain the constraints from the previous model and add an additional constraint. For strong or scalar invariance we add the constraint that the intercepts to are equal across groups. In the code below, **mcons** stands for the measurement constant, i.e., the intercept.

```
. sem (F1 -> read write math science) ///
      (F1 -> read@1)                   ///  /* set loadings  to 1 in both groups */
      (read <- F1 _cons@0),            ///  /* set intercept to 0 in both groups */
      group(female)                    ///
      ginvariant(mcoef mcons)          ///
      mean(F1) variance(e.read@0)

[Output Redacted]

LR test of model vs. saturated: chi2(12)  =    118.27, Prob > chi2 = 0.0000
```

Model 4: Strict Invariance (strong invariance plus equal residuals)

For strict invariance, we add the constraint that the residual variances are to be equal across groups. We do not need to use the **ginvariant** option because constraing the residual variances to be equal implies the the loadings and intercepts are also equal.

```
. sem (F1 -> read write math science)  ///
      (F1 -> read@1)                    ///  /* set loadings  to 1 in both groups */
      (read <- F1 _cons@0),             ///  /* set intercept to 0 in both groups */
      group(female)                     ///
      var(e.read@0) var(e.write@v2)     ///  /* set residuals to be equal */
      var(e.math@v3) var(e.science@v4)  ///  /* set residuals to be equal */
      mean(F1)

[Output Redacted]

LR test of model vs. saturated: chi2(15)  =    128.03, Prob > chi2 = 0.0000
```

Model 5: Strict Invariance Plus Factor Means

Next, we constrain the factor means to be equal across groups in addition to all previous constraints.

```
. sem (F1 -> read write math science)   ///
      (F1 -> read@1)                     ///  /* set loadings  to 1 in both groups */
      (read <- F1 _cons@0),              ///  /* set intercept to 0 in both groups */
      group(female)                      ///
      var(e.read@0)  var(e.write@v2)     ///
      var(e.math@v3) var(e.science@v4)   ///
      mean(F1@m1)                             /* set factor means to be equal */

[Output Redacted]

LR test of model vs. saturated: chi2(16)  =    128.59, Prob > chi2 = 0.0000
```

Model 6: Strict Invariance Plus Factor Means & Factor Variances

Add one last constraint to our list: Equal factor variances.

```
. sem (F1 -> read write math science)      ///
      (0: F1 -> read@1)                    ///  /* set loading  to 1 in group 0 */
      (1: F1 -> read@1)                    ///  /* set loading  to 1 in group 1 */
      (read <- F1 _cons@0),                ///  /* set intercept to 0 in both groups */
      group(female)                        ///  /* multiple-group analysis   */
      var(e.read@0)  var(e.write@v2)       ///  /* set residuals to be equal */
      var(e.math@v3) var(e.science@v4)     ///  /* set residuals to be equal */
      var(F1@v5)                           ///  /* set factor variances to be equal */
      mean(F1@m1)                               /* set factor means to be equal */

[Ouput Redacted]

LR test of model vs. saturated: chi2(17)  =    128.77, Prob > chi2 = 0.0000
```

Summary Table

For the purposes of this presentation, we are just going to look at the chi-square values and degrees of freedom for each model. In actual practice, we would want to also look at measures like RMSEA and CFI.

We start off by comparing the metric invariance model to the one in which all parameters are free to vary. The metric invariance model does not fit significantly worse. However, when we compare the strong invariance with metric invariance, the fit does become significantly worse. Even though we didn't meet the strong factor invariance level, we show how the remaining comparisons are computed.

model	chi2	df	ref model	delta chi2	delta df	P
Model 1 free all parameters	84.53	6	-			
Model 2 metric (loadings)	89.02	9	1	4.49	3	.2132
Model 3 strong (intercepts)	118.27	12	2	29.25	3	.0000
Model 4 strict (residuals)	128.03	15	3	9.76	3	.0207
Model 5 plus factor means	128.59	16	4	.56	1	.4543
Model 6 plus factor var	128.77	17	5	.18	1	.6714

Partial Strong Invariance

We did not achieve full strong factor invariance. However, inspection of the the model with all parameters free to vary suggests that the problem may lie with the intercepts for the variable write. Let's try a model with partial strong invariance, in which, the intercepts for math and science are constrained to be equal but are allowed to vary for write.

```
sem (F1 -> read write math science) ///
    (F1 -> read@1)                  ///  /* set loadings  to 1 in both groups    */
    (read <- F1 _cons@0)            ///  /* set intercept to 0 in both groups    */
    (0: write <- F1 _cons@i1)       ///  /* allow intercepts for write to differ */
    (1: write <- F1 _cons@i2),      ///  /* allow intercepts for write to differ */
    group(female)                   ///
    ginvariant(mcoef mcons)         ///
    variance(e.read@0)              ///  /* set residual to 0     */
    mean(F1)                             /* estimate factor means */

[Output Redacted]
```

```
LR test of model vs. saturated: chi2(11)  =     91.99, Prob > chi2 = 0.0000
```

Partial Strict Invariance

If we free the residual variances for write, we get a model for partial strict invariance.

```
sem (F1 -> read write math science)  ///
    (0: F1 -> read@1)                ///  /* set loadings  to 1 in both groups */
    (1: F1 -> read@1)               ///
    (read <- F1 _cons@0)             ///  /* set intercept to 0 in both groups */
    (0: write <- F1 _cons@i1)        ///  /* allow intercepts for write to differ */
    (1: write <- F1 _cons@i2),       ///  /* allow intercepts for write to differ */
    group(female)                    ///
    ginvariant(mcoef mcons)          ///
    variance(e.read@0)               ///  /* set residual to 0 */
    var(e.math@v3) var(e.science@v4) ///
    mean(F1)

[Output Redacted]

LR test of model vs. saturated: chi2(13)  =     96.03, Prob > chi2 = 0.0000
```

Now that the intercepts and residual variances for write are allowed to vary across groups, we can run models constrain factor means and factor variances to be equal (code not shown). Here is the summary table for the partial invariance models.

Revised Summary Table for Partial Invariance

model	chi2	df	ref model	delta chi2	delta df	P
1 free all parameters	84.53	6	-			
2 metric (loadings)	89.02	9	1	4.49	3	.2132
3 partial strong	91.99	11	2	2.97	2	.2265
4 partial strict	96.03	13	3	4.04	2	.1327
5 plus factor means	96.59	14	4	.53	1	.4666
6 plus factor var	96.77	15	5	.21	1	.6468

You will note that model 6 (partial strict plus factor means and variances) does not fit significantly worse than model 1 (all parameters free).

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