Modeling Interactions in Count Data Regression
Principles and Implementation in `Stata`

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Definition of interactions

- "By interactions we mean an interplay among predictors that produces an effect on the outcome Y that is *different from the sum of the effects of the individual predictors*." (Cohen et al. 2003, 255)
- "Two explanatory variables are said to interact in determining a response variable when *the partial effect of one depends on the value of the other*." (Fox 2008, 131)

$\rightarrow$ From an analytical point of view, an interaction effect can be defined *as the marginal effect of a marginal effect*

Identification of interaction effects in the linear model

- Linear model with interaction term x_1x_2:

$$E(y|\mathbf{x}) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_M x_1 x_2 + \sum_{j=3}^{k} \beta_j x_j \tag{1}$$

- Interaction effects (if x_j is dichotomous, then $x_j = d_j$):

$$\frac{\partial^2 E(y|\mathbf{x})}{\partial x_1 \partial x_2} = \frac{\partial \Delta E(y|\mathbf{x})}{\partial x_1 \Delta d_2} = \frac{\Delta^2 E(y|\mathbf{x})}{\Delta d_1 \Delta d_2} = \beta_M \tag{2}$$

$\rightarrow$ In the linear model, the interaction effect is in any case equal to the product term coefficient β_M
Significance testing: Wald-test for β_M

Interaction effects in nonlinear models

Current state of research

- ... in Logit & Probit models (Ai & Norton 2003; Berry et al. 2010; Bowen 2012; Greene 2010; Karaca-Mandic et al. 2012; Norton et al. 2004; Seymour 2011)
- ... within the GLM framework (Tsai & Gill 2013)
- To date, no explicit contributions covering the identification of interaction effects in count data models are available

Characteristics of Interaction effects in nonlinear models

- In contrast to the linear model (see Eq. (2)), the interaction effect does not equal β_M
- A significant interaction effect is possible even when $\beta_M = 0$ (model inherent interaction effect)
  $\rightarrow$ Statistical significance cannot be tested by applying a Wald-test for β_M
- The interaction effect is dependent on covariates and thus subject to variation across individuals
- The interaction effect may have different signs for different individuals
  $\rightarrow$ The sign of β_M does not necessarily indicate the direction of the interaction effect
- The *total interaction effect* is composed additively of a *model inherent interaction effect* and a *product term induced interaction effect*

- Inverted link function:

$$E(y|\mathbf{x}) = \exp(\boldsymbol{x\beta}) = \mu \tag{3}$$

- Poisson model (stochastic component)

$$f(y|\mu) = Pr(Y = y) = \frac{\exp(-\mu)\mu^y}{y!}; y = 0, 1, 2, \ldots; \mu > 0 \tag{4}$$

- Negative binomial model (stochastic component)

$$f(y|\mu,\alpha) = Pr(Y = y) = \frac{\Gamma(y+\alpha^{-1})}{\Gamma(y+1)\Gamma(\alpha^{-1})}\left(\frac{\alpha^{-1}}{\alpha^{-1}+\mu}\right)^{\alpha^{-1}} \left(\frac{\mu}{\alpha^{-1}+\mu}\right)^{y}; y = 0, 1, 2, \ldots; \mu > 0; \alpha \geq 0 \tag{5}$$

- Count data model with interaction term $x_1 x_2$:

$$E(y|\mathbf{x}) = \exp\left(\beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_M x_1 x_2 + \sum_{j=3}^{k} \beta_j x_j\right) \qquad (6)$$

- Total interaction effect (ι_t)

$$\iota_t = \frac{\partial^2 E(y|\mathbf{x})}{\partial x_1 \partial x_2} = \left[\left(\beta_1 + \beta_M x_2\right)\left(\beta_2 + \beta_M x_1\right) + \beta_M\right] E(y|\mathbf{x}) \qquad (7)$$

- Rearranging terms uncovers the model inherent (ι_m) and the product term induced (ι_p) interaction effect

$$\iota_t = \underbrace{\beta_1 \beta_2 E(y|\mathbf{x})}_{\iota_m} + \underbrace{\beta_M \left(\beta_1 x_1 + \beta_2 x_2 + \beta_M x_1 x_2 + 1\right) E(y|\mathbf{x})}_{\iota_p} \qquad (8)$$

Delta method standard errors

According to Ai & Norton (2003), standard errors for the interaction effects can be obtained by applying the Delta method for variance estimation

- total interaction effect

$$\hat{\sigma}^2_{\iota_t} = \left(\frac{\partial \iota_t}{\partial \boldsymbol{\beta}}\right) \hat{\mathbf{V}} \left(\frac{\partial \iota_t}{\partial \boldsymbol{\beta}}\right)' \tag{9}$$

- model inherent interaction effect

$$\hat{\sigma}^2_{\iota_m} = \left(\frac{\partial \iota_m}{\partial \boldsymbol{\beta}}\right) \hat{\mathbf{V}} \left(\frac{\partial \iota_m}{\partial \boldsymbol{\beta}}\right)' \tag{10}$$

- product term induced interaction effect

$$\hat{\sigma}^2_{\iota_p} = \left(\frac{\partial \iota_p}{\partial \boldsymbol{\beta}}\right) \hat{\mathbf{V}} \left(\frac{\partial \iota_p}{\partial \boldsymbol{\beta}}\right)' \tag{11}$$

Example with artificial data ($\beta_1 < 0; \beta_2 > 0; \beta_M > 0$)

- Simulation of a Poisson model with $\eta = -6 - 2x_1 + 2x_2 + .5x_1x_2$
  $x_1, x_2 \sim N(0;1)$; $n = 10.000$

- Estimation results (`poisson` command)

variable	**coef.**	**se**	**p**
constant	-6.148	.172	$<.001$
x_1	-2.038	.073	$<.001$
x_2	1.990	.086	$<.001$
x_1x_2	.493	.042	$<.001$

$LL = -935.011$; LR-Test (Nullmodell): $\chi^2 = 1,424.69$; $df = 3$; $p < .001$; $PseudoR^2 = .432$; $AIC = 1,878.022$; $BIC = 1,906.82$

- Calculation of interaction effects and standard errors via `predictnl` command

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Calculation of $\iota_t, \iota_m, \iota_p$ & standard errors with `predictnl`

- Estimate Poisson model

```
poisson y x1 x2 x1x2
```

- Calculate predicted count

```
predict expcount
```

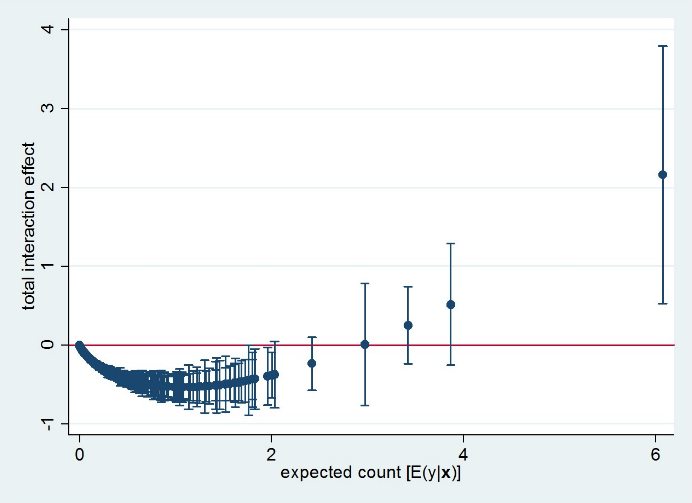
- Calculate total, model inherent & product term induced interaction effects and corresponding standard errors

```
predictnl total = ((_b[x1] + _b[x1x2]*x2)*(_b[x2] +
_b[x1x2]*x1) + _b[x1x2])*expcount, se(setotal)
```

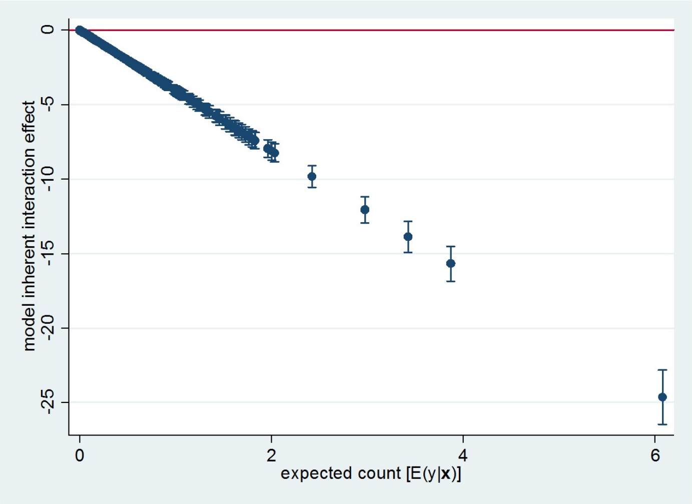
```
predictnl inherent = _b[x1]*b[x2]*expcount,
se(seinherent)
```

```
nlpredict product = _b[x1x2]*(_b[x1]*x1 + _b[x2]*x2 +
_b[x1x2]*x1*x2 + 1)*expcount, se(seproduct)
```

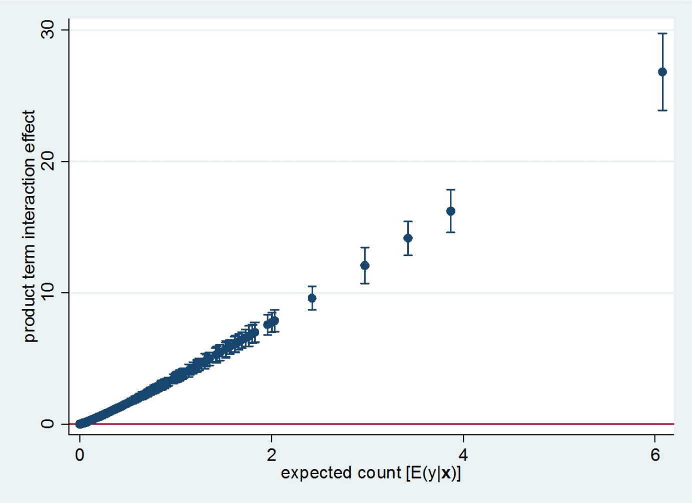
Total interaction effect



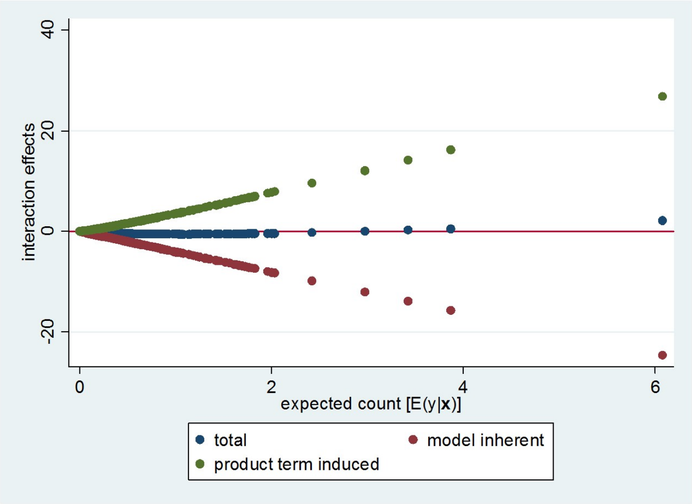
Model interent interaction effect



Product-term induced interaction effect



All interaction effects



Next steps

- Calculate average interaction effects & corresponding standard errors (analogous to AMEs)
- Calculate interaction effects & corresponding standard errors for dichotomous covariates
- Allow for more than one two-way and for three-way interactions?!?
- Put all these features into a `Stata` program
- Simulate distributions of interaction effects from a theoretical perspective (e.g. exploring the relevance of ι_m)
  $\rightarrow$ Learn how to adequately interpret these interaction effects in nonlinear models

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Reference list can be requested via email