Modeling Interactions in Count Data Regression
Principles and Implementation in Stata

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"By interactions we mean an interplay among predictors that produces an effect on the outcome $Y$ that is different from the sum of the effects of the individual predictors." (Cohen et al. 2003, 255)

"Two explanatory variables are said to interact in determining a response variable when the partial effect of one depends on the value of the other." (Fox 2008, 131)

→ From an analytical point of view, an interaction effect can be defined as the marginal effect of a marginal effect.
Identification of interaction effects in the linear model

- Linear model with interaction term $x_1 x_2$:

$$E(y|x) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_M x_1 x_2 + \sum_{j=3}^{k} \beta_j x_j \quad (1)$$

- Interaction effects (if $x_j$ is dichotomous, then $x_j = d_j$):

$$\frac{\partial^2 E(y|x)}{\partial x_1 \partial x_2} = \frac{\partial \Delta E(y|x)}{\partial x_1 \Delta d_2} = \frac{\Delta^2 E(y|x)}{\Delta d_1 \Delta d_2} = \beta_M \quad (2)$$

→ In the linear model, the interaction effect is in any case equal to the product term coefficient $\beta_M$

*Significance testing*: Wald-test for $\beta_M$
Interaction effects in nonlinear models

Current state of research


- ... within the GLM framework (Tsai & Gill 2013)

- To date, no explicit contributions covering the identification of interaction effects in count data models are available
In contrast to the linear model (see Eq. (2)), the interaction effect does not equal $\beta_M$.

A significant interaction effect is possible even when $\beta_M = 0$ (model inherent interaction effect).

→ Statistical significance cannot be tested by applying a Wald-test for $\beta_M$.

The interaction effect is dependent on covariates and thus subject to variation across individuals.

The interaction effect may have different signs for different individuals.

→ The sign of $\beta_M$ does not necessarily indicate the direction of the interaction effect.

The total interaction effect is composed additively of a model inherent interaction effect and a product term induced interaction effect.
Introduction to count data models

- Inverted link function:

\[ E(y|x) = \exp(x\beta) = \mu \]  

- Poisson model (stochastic component)

\[ f(y|\mu) = Pr(Y = y) = \frac{\exp(-\mu)\mu^y}{y!}; y = 0, 1, 2, \ldots; \mu > 0 \]  

- Negative binomial model (stochastic component)

\[ f(y|\mu, \alpha) = Pr(Y = y) = \frac{\Gamma(y + \alpha^{-1})}{\Gamma(y + 1)\Gamma(\alpha^{-1})} \left( \frac{\alpha^{-1}}{\alpha^{-1} + \mu} \right)^{\alpha^{-1}} \left( \frac{\mu}{\alpha^{-1} + \mu} \right)^y; y = 0, 1, 2, \ldots; \mu > 0; \alpha \geq 0 \]
Interaction effects in count data models

- Count data model with interaction term $x_1x_2$:

$$E(y|x) = \exp \left( \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_M x_1 x_2 + \sum_{j=3}^{k} \beta_j x_j \right)$$  \hspace{1cm} (6)

- Total interaction effect ($\iota_t$)

$$\iota_t = \frac{\partial^2 E(y|x)}{\partial x_1 \partial x_2} = \left[ (\beta_1 + \beta_M x_2) (\beta_2 + \beta_M x_1) + \beta_M \right] E(y|x)$$ \hspace{1cm} (7)

- Rearranging terms uncovers the model inherent ($\iota_m$) and the product term induced ($\iota_p$) interaction effect

$$\iota_t = \underbrace{\beta_1 \beta_2 E(y|x)}_{\iota_m} + \underbrace{\beta_M (\beta_1 x_1 + \beta_2 x_2 + \beta_M x_1 x_2 + 1)}_{\iota_p} E(y|x)$$ \hspace{1cm} (8)
According to Ai & Norton (2003), standard errors for the interaction effects can be obtained by applying the Delta method for variance estimation:

- **total interaction effect**
  \[
  \hat{\sigma}_{\iota t}^2 = \left( \frac{\partial \iota_t}{\partial \beta} \right)^\prime \hat{\mathbf{V}} \left( \frac{\partial \iota_t}{\partial \beta} \right) \tag{9}
  \]

- **model inherent interaction effect**
  \[
  \hat{\sigma}_{\iota m}^2 = \left( \frac{\partial \iota_m}{\partial \beta} \right)^\prime \hat{\mathbf{V}} \left( \frac{\partial \iota_m}{\partial \beta} \right) \tag{10}
  \]

- **product term induced interaction effect**
  \[
  \hat{\sigma}_{\iota p}^2 = \left( \frac{\partial \iota_p}{\partial \beta} \right)^\prime \hat{\mathbf{V}} \left( \frac{\partial \iota_p}{\partial \beta} \right) \tag{11}
  \]
Example with artificial data ($\beta_1 < 0; \beta_2 > 0; \beta_M > 0$)

- Simulation of a Poisson model with $\eta = -6 - 2x_1 + 2x_2 + .5x_1x_2$
  $x_1, x_2 \sim N(0; 1); n = 10,000$

- Estimation results (poisson command)

<table>
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<th>coef.</th>
<th>se</th>
<th>p</th>
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<td>$x_1x_2$</td>
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<td>.042</td>
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</tr>
</tbody>
</table>

$LL = -935.011$; LR-Test (Nullmodell): $\chi^2 = 1,424.69; df = 3; p < .001$; $PseudoR^2 = .432$; $AIC = 1,878.022$; $BIC = 1,906.82$

- Calculation of interaction effects and standard errors via predictnl command

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Interactions in Count Data Models
Hamburg, June 2014
Calculation of $\ell_t, \ell_m, \ell_p$ & standard errors with `predictnl`

- Estimate Poisson model
  ```stata```
  poisson y x1 x2 x1x2
  ```
- Calculate predicted count
  ```stata```
  predict expcount
  ```
- Calculate total, model inherent & product term induced interaction effects and corresponding standard errors
  ```stata```
  predictnl total = ((_b[x1] + _b[x1x2]*x2)*(_b[x2] + _b[x1x2]*x1) + _b[x1x2])*expcount, se(setotal)
  ```stata```
  predictnl inherent = _b[x1]*b[x2]*expcount, se(seinherent)
  ```stata```
  nlpredict product = _b[x1x2]*(_b[x1]*x1 + _b[x2]*x2 + _b[x1x2]*x1*x2 + 1)*expcount, se(seproduct)
Total interaction effect
Model interaction interaction effect
Product-term induced interaction effect

[Graph showing the relationship between expected count $E(y|x)$ and product term interaction effect.]
All interaction effects
Next steps

- Calculate average interaction effects & corresponding standard errors (analogous to AMEs)
- Calculate interaction effects & corresponding standard errors for dichotomous covariates
- Allow for more than one two-way and for three-way interactions?!?
- Put all these features into a Stata program
- Simulate distributions of interaction effects from a theoretical perspective (e.g. exploring the relevance of \( \iota_m \)) → Learn how to adequately interpret these interaction effects in nonlinear models
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Reference list can be requested via email