Bootstrap LM Tests for the Box Cox Tobit Model

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Introduction

- This presentation sets out a specification test of the Tobit model against the alternative of a specification described by the Box Cox transformation.
- An LM test is used to test the null hypothesis of no specification error as this requires estimates of the restricted (nested) Tobit) model
- The size and power of the test using asymptotic and bootstrap critical values is estimated by the empirical rejection probabilities for small sample sizes

1. The Box Cox Tobit Model

- The Tobit model is used to address censoring and corner solution problems.
- When censoring occurs at zero, the model in both applications is written:

$$y_i^* = x_i^{'}\beta + \epsilon_i, \quad i = 1, .., N \qquad (1)$$

where y_i^* is a \`latent' variable and $\epsilon_i \sim NID\left(0, \sigma^2\right)$. The observation rule is:

$$y_i = \begin{cases} y_i^* & \text{if } y_i^* \geq 0 \\ 0 & \text{if } y_i^* < 0 \end{cases}$$

- In censored data problems, we are usually interested in the features of y_i^* such as $E\left[y_i^* \mid x_i\right]$. For corner solutions however, it is $E\left[y_i \mid x_i\right]$ that is of interest.
- Estimation of the parameters β, and σ in (1) is by Maximum Likelihood (ML), with individual contribution to the log-likelihood given by:

$$\ln L_i = d_i \ln\left[\frac{1}{\sigma}\phi\left(\frac{y_i - x_i^{'}\beta}{\sigma}\right)\right] + (1 - d_i)\ln\left[1 - \Phi\left(\frac{x_i^{'}\beta)}{\sigma}\right)\right]$$

1. The Box Cox Tobit Model

- As Moffat (2003) noted however, there are many instances where y_i exhibits positive skew that cannot be attributed to the asymmetric censoring.
- In the double hurdle model, Moffat takes the following transformation of y_i to preserve normality:

$$y_i^T = \frac{y_i^\lambda - 1}{\lambda} \qquad 0 \cdot \ \lambda \cdot \ 1$$

- The transformation, originally proposed by Box & Cox (1966) for uncensored data, was designed to ensure that the model for y_i^T is:

  1. Linear in the explanatory variables
  2. Has a constant conditional error variance $E[\epsilon\epsilon^{'} \mid X] = \sigma^2 I_N$
  3. Has a normally distributed error term

- The above properties are essential for the ML-estimators to be consistent for the true parameters in the Tobit model (1): $\hat{\beta} \xrightarrow{p} \beta \quad \hat{\sigma} \xrightarrow{p} \sigma$

1. The Box Cox Tobit Model

- Applying the Box Cox Transformation (BCT) to the Tobit model therefore, leads to the following observation rule:

$$y_i^T = \begin{cases} y_i^{T*} & \text{if } y_i^{T*} \geq -1/\lambda \\ -1/\lambda & \text{if } y_i^{T*} < -1/\lambda \end{cases}$$

- where y_i^T* is the `transformed' latent variable with specification:

$$y_i^{*T} = x_i^{'}\beta + \epsilon_i, \qquad \epsilon_i \sim NID\left(0, \sigma^2\right)$$

- This should now satisfy (or approximately) the distributional requirements for the ML-estimator to be consistent.
- By a change of variables, the i^{th} contribution to the log-likelihood is:

$$\ln L_i = d_i \ln \left[\frac{y_i^{\lambda-1}}{\sigma} \phi \left(\frac{\left(y_i^{\lambda} - 1\right)/\lambda - x_i^{'}\beta}{\sigma} \right) \right] + (1 - d_i) \ln \left[1 - \Phi \left(\frac{1/\lambda + x_i^{'}\beta}{\sigma} \right) \right] \quad (2)$$

2. LM test of the Tobit specification

- A test of the linearity, homoskedasticity and normality assumptions of the Tobit specification, is therefore equivalent to a test of:

$$H_0 : \lambda = 1$$

- against the more general alternative:

$$H_1 : \lambda \neq 1$$

- The LM-statistic is the easiest to compute as this requires parameter estimates under the restrictions imposed by the null $\widetilde{\theta} = (\widetilde{\beta}, \widetilde{\sigma}, 1)$.
- Denoting τ as an $N \times 1$ vector one 1's, $\widetilde{G} = (\widetilde{g}_1, ..., \widetilde{g}_N)'$ where $\widetilde{g}_i = \frac{\partial \ln L_i}{\partial \theta}|_{\widetilde{\theta}}$ represents the i^{th} contribution to the unrestricted score evaluated at the restricted $\widetilde{\theta}$, then the OPG-version of the LM-test is:

$$LM = \tau' \widetilde{G} (\widetilde{G}' \widetilde{G}')^{-1} \widetilde{G}' \tau \quad \xrightarrow{d} \chi_1^2$$

2. LM test of the Tobit specification

- In this form, the LM-statistic is simply $N \times R_u^2$ from artificial regression:

$$1 = \tilde{g}_i' \pi + e_i$$

- From (2), the individual elements of $\tilde{g}_i$ are:

$$\frac{\partial \ln L_i(\theta)}{\partial \beta} |_{\tilde{\theta}} = d_i \frac{\tilde{v}_{i1}}{\tilde{\sigma}^2} x_i + (1 - d_i) \frac{-\phi(\tilde{v}_{i2}/\tilde{\sigma})}{1 - \Phi(\tilde{v}_{i2}/\tilde{\sigma})} \frac{x_i}{\tilde{\sigma}} \quad (3)$$

$$\frac{\partial \ln L_i(\theta)}{\partial \sigma} |_{\tilde{\theta}} = d_i \frac{1}{\tilde{\sigma}} \left[\frac{v_{i1}^2}{\tilde{\sigma}^2} - 1 \right] + (1 - d_i) \frac{\phi(\tilde{v}_{i2}/\tilde{\sigma})}{1 - \Phi(\tilde{v}_{i2}/\tilde{\sigma})} \frac{\tilde{v}_{i2}}{\tilde{\sigma}^2} \quad (4)$$

$$\frac{\partial \ln L_i(\theta)}{\partial \lambda} |_{\tilde{\theta}} = d_i \left[\ln y_i - \frac{\tilde{v}_{i1}}{\tilde{\sigma}^2} [y_i (\ln y_i - 1) + 1] \right] + (1 - d_i) \frac{\phi(\tilde{v}_{i2}/\tilde{\sigma})}{1 - \Phi(\tilde{v}_{i2}/\tilde{\sigma})} \frac{1}{\tilde{\sigma}} \quad (5)$$

- where $\tilde{v}_{i1} = y_i - (1 + x_i' \tilde{\beta})$ and $\tilde{v}_{i2} = 1 + x_i' \tilde{\beta}$. Under the restrictions imposed by the null, (3) and (4) are the scores of the Tobit model evaluated at the Tobit MLE's; (5) can therefore be constructed from these estimates.

3. Bootstrap Critical Values

- The critical value for a test of size- α is the solution to $G_n(c_{n,\alpha}; F_0) = 1 - \alpha$ where $G_n(c; F_0) = Pr(LM \cdot \ c)$ and $F_0 = F(x_i, y_i; \theta_0)$ is the distribution of the data.
- Unless F_0 is known, $c_{n,\alpha}$ cannot be obtained and we use critical values from the limiting distribution under H_0, i.e.: $G_\infty(c_{\infty,\alpha}) = Pr(\chi^2_1 \cdot \ c_{\infty,\alpha}) = 1 - \alpha$
- The size of the test using $c_{\infty,\alpha}$ is $\alpha + O(n^{-1})$ which can be determined through the asymptotic expansion $G_n(c; F_0) = G_\infty(c) + O(n^{-1})$. This error can be large
- An alternative approach is to obtain critical values from the bootstrap null distribution $G_n(c; F_n)$ which replaces F_0 with a consistent estimator F_n. Then:

$$G_n(c; F_0) = G_n(c; F_n) + O(n^{-3/2}) \qquad (6)$$

- which has a smaller error of order $O(n^{-3/2})$. The critical value $c^\dagger_{n,\alpha}$ solving $G_n(c^\dagger_{n,\alpha}; F_n) = 1 - \alpha$ be found by Monte Carlo simulation as the $1 - \alpha$ quantile of the B ordered *bootstrap statistics* $LM^\dagger_1, ..., LM^\dagger_B$

4. The Parametric Bootstrap Algorithm

- The null $H_0 : \lambda = 1$ is rejected if $LM > c^{\dagger}_{n,\alpha}$
- In the B - simulations, each bootstrap sample is generated by re-sampling x_i from the EDF, while generating y_i from $F(y_i, | \, x_i; \tilde{\theta})$. The algorithm is:

1. Estimate the Tobit model parameters: $\hat{\beta}$, $\hat{\sigma}$ .This imposes the constraint $\lambda = 1$
2. Draw a random sample of size N from the EDF of x_i and denote these $x^{\dagger}_i, ..., x^{\dagger}_n$
3. Generate N errors from $N(0, \hat{\sigma}^2)$ and denote these $\epsilon^{\dagger}_1, ..., \epsilon^{\dagger}_n$
4. Use the values in steps 2 and 3 to generate a bootstrap sample of size N $y^{*\dagger}_i = x^{\dagger\prime}_i \hat{\beta} + \epsilon^{\dagger}_i$ and compute $y^{\dagger}_i = \max(0, y^{*\dagger}_i)$
5. Estimate the Tobit model using the bootstrap sample and compute the contributions to the scores $\tilde{g}^{\dagger}_i, ..., \tilde{g}^{\dagger}_N$
6. Estimate the artificial regression $1 = \tilde{g}^{\dagger\prime}_i \delta + u_i$ and compute $LM^{\dagger}_b = N \times R^2_u$
7. Repeat steps 2 – 6 a total of B- times and compute the critical value $c^{\dagger}_{n,\alpha}$ as the $1 - \alpha$ percentile of the B ordered bootstrap LM-test statistics.

5. Monte-Carlo Design

- The size and power of the LM-test using bootstrap and first-order asymptotic critical values can be estimated from the empirical rejection probabilities.
- The data for the Monte-Carlo experiments is generated from the DGP:

$$y_i^{*T} = x_i^{'}\beta + \epsilon_i, \qquad y_i^T = \begin{cases} y_i^{T*} & \text{if } y_i^{T*} \geq -1/\lambda \\ -1/\lambda & \text{if } y_i^{T*} < -1/\lambda \end{cases}$$

$$y_i = \left(\lambda y_i^{T*} + 1\right)^{1/\lambda}$$

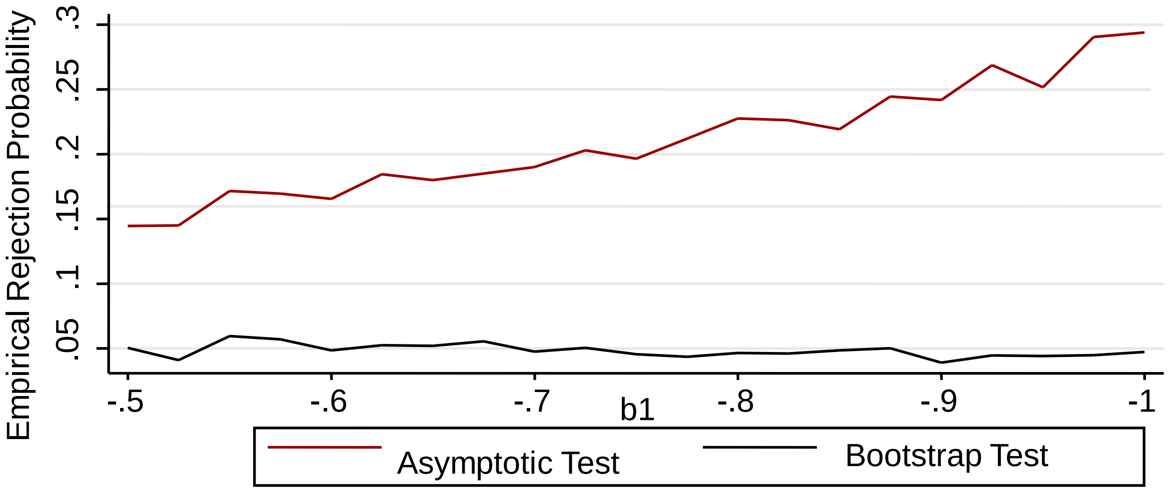
The experiments consist of the following steps:

1. Generate N values for ϵ_i and x_i from a specified DGP and compute $y_i^{*T}, y_i^T \; y_i$
2. Estimate the LM statistic for testing $H_0 : \lambda = 1$ as detailed earlier
3. Compute the bootstrap critical value at the α -level for testing $H_0 : \lambda = 1$
4. Repeat steps 1-3, T - times and count the rejections R. The empirical rejection probability R/T , is an estimate of the true rejection probability p.

- As $R \sim B(T,p)$, then $\sqrt{T}\,(R/T - p) \xrightarrow{d} N\left[0, p(1-p)\right]$. Thus for $p = 0.05$ and $T = 2000$, $Pr(0.04 \cdot \; R/T \cdot \; 0.06 \mid p = 0.05) \approx 0.95$

5.1 Size Estimates

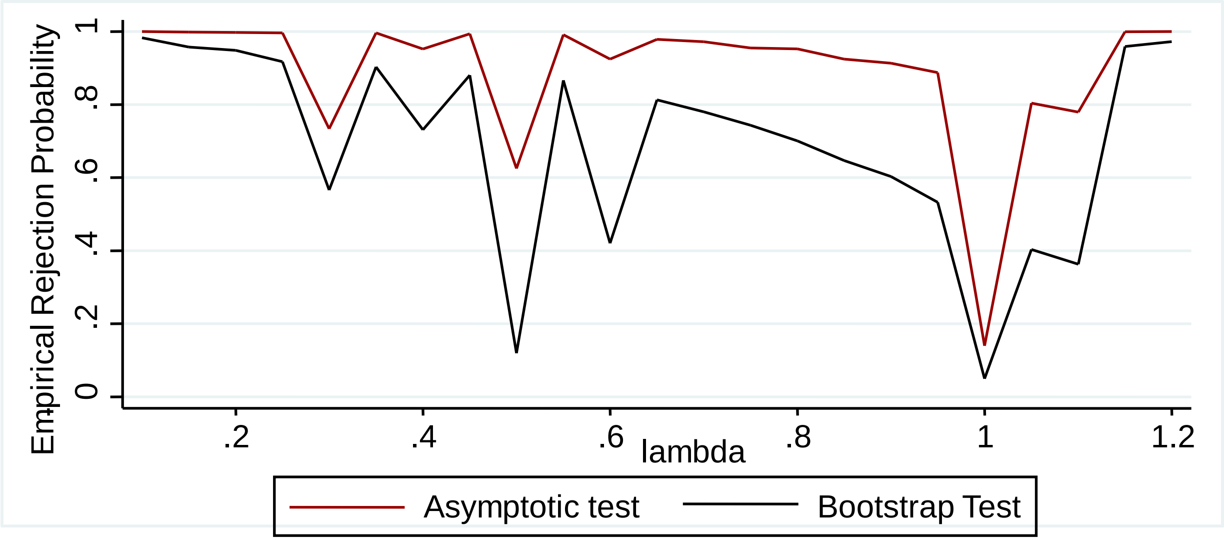
- Under $H_0 : \lambda = 1$, the empirical rejection probability is an estimate of the size of the LM-test using bootstrap & asymptotic critical values .
- For these experiments $N = 25$, $\alpha = 0.05$, $B = 499$, $T = 2000$, $\epsilon_i \sim NID(0,1)$ and $x_i^{'}\beta = \beta_0 + \beta_1 x_{i1}$ where: $\ln x_i \sim N(1, 0.5)$, $\beta_0 = 1$ and $\beta_1 \in \{-.5, -.55, -.6, -.65, -.7, -.75, -.8, -.85, -.9, -.95\}$. The size estimates are:



- Using bootstrap critical values there is no size distortion. This is not the case using asymptotic critical values which result in large size distortions

5.2 Power Estimates (1)

- Under $H_1 : \lambda = \lambda_1$, the empirical rejection probability is an estimate of the power of the LM-test against the alternative.
- For these experiments, $N = 25$, $\alpha = 0.05$, $B = 499$, $T = 2000$, $\epsilon_i \sim NID(0,1)$ and $x_i' \beta = \beta_0 + \beta_1 x_{i1}$ where $\ln x_i \sim N(1, 0.5)$, $\beta_0 = 1$, $\beta_1 = -.5$ and $\lambda = \lambda_1 \in \{.1, .15, .2, ..., 1.3\}$. The power estimates are:



- With the exception of $\lambda = 0.5$, the LM-test using bootstrap critical values at the 5% level of significance seems reasonably powerful for $N = 25$

5.3 Power Estimates (2)

- Whilst the LM-test exhibits reasonable power for $\lambda \neq 1$, it is worth examining the power against DGP's where a $\lambda \neq 1$ would necessary for consistency
- For these experiments, $N = 100$, $\alpha = 0.05$, $B = 499$, $T = 2000$, and the data are generated using similar DGP's to those used by Drukker(2002):

$$y_i^* = 1 + x_{i1} + x_{i2} + x_{i3} + \epsilon_i \sqrt{h(z_i^{'}\alpha)},$$

$$x_{i1} \sim N(0,1) \qquad x_{i2} = .3x_{1i} + u_{i2}, \; u_{i2} \sim N(0,1)$$

$$x_{i3} = .3x_{1i} + u_{i3}, \;\; u_{i3} \sim N(0,1)$$

- The ϵ_i are generated from, $N(0,1)$, t_4, and χ_5^2, distributions and the function $h(z_i^{'}\alpha) = 1$ for homoskedastic and $h(z_i^{'}\alpha) = e^{2x_{i1}}$ for hetroskedastic errors.
- The following table sets out the power estimates:

Distribution	$h(z_i^{'}\alpha) = 1$	$h(z_i^{'}\alpha) = e^{2x_{i1}}$
$N(0,1)$	N/A	0.734
t_4	0.085	0.795
χ_5^2	0.140	0.872

6. Description of `bctobit' Program

```
bctobit [, Fixed Nodots bfile(string) reps(integer 499)]
```

Description

- bctobit computes the LM-statistic for testing $H_0 : \lambda = 1$ against $H_1 : \lambda \neq 1$ in the Box Cox Tobit model. This is equivalent to testing the linearity, normality and homoskedasticity assumptions of the Tobit specification.
- The regressors are assumed to be random, and critical values are obtained from the bootstrap null distribution of the LM test statistic by repeated sampling from the (parametric) bootstrap DGP.

Options

- Fixed - specifies that the regressors are fixed in the bootstrap null distribution
- Nodots – suppresses the replication dots
- bfile(name) – the name of the saved file which contains the LM-statistics computed from the bootstrap samples
- reps(#) - the number of samples to be drawn from the bootstrap DGP to estimate the percentiles of the bootstrap null distribution. Default is 499

6. Description of `bctobit' Program

```
Tobit regression                                  Number of obs   =        100
                                                  LR chi2(3)      =     139.54
                                                  Prob > chi2     =     0.0000
Log likelihood = -117.08451                       Pseudo R2       =     0.3734

------------------------------------------------------------------------------
           y |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
          x1 |   .8808724   .1447619     6.08   0.000     .5935602    1.168185
          x2 |   .9554311   .1253373     7.62   0.000     .7066713    1.204191
          x3 |   .9387104   .1204485     7.79   0.000     .6996535    1.177767
       _cons |   1.200638   .1305344     9.20   0.000     .9415631    1.459712
-------------+----------------------------------------------------------------
      /sigma |    1.05923   .0898169                      .8809688    1.237492
------------------------------------------------------------------------------
  Obs. summary:         29  left-censored observations at y<=0
                        71     uncensored observations
                         0 right-censored observations

. bctobit, reps(299)
Bootstrap replications (299)
----+--- 1 ---+--- 2 ---+--- 3 ---+--- 4 ---+--- 5
..................................................    50
..................................................   100
..................................................   150
..................................................   200
..................................................   250
.................................................
LM test of Tobit specification
                   Bootstrap critical values
    lm         %10         %5        %1
   1.4669    2.86527   4.1014972 10.135839
```

7. Further Research....

- A natural extension would be to consider the alternative of a Box Cox transformation with an error term that is hetroskedastic

$$y_i^{T*} = x_i^{'}\beta + \epsilon_i\sqrt{h(z_i^{'}\alpha)},$$

- where h is an unknown function , with $h'(.) \neq 0$, $h(0) = 1$ and $h'(0) = \kappa$
- A test of the joint hypothesis: $H_1 : \lambda = 1,\ \ \eta = 0$ against the alternaitve of $H_1 : \lambda \neq 1,\ \ \eta \neq 0$ is equivalent to testing the validity of the Tobit specification.
- The LM statistic would now be based on the additional components of the score vector, evaluated at the restrictions given by the null. These are:

$$\frac{\partial \ln L_i(\theta)}{\partial \alpha} \,|_{\widetilde{\theta}} = d_i \frac{1}{2}\left[\frac{\widetilde{v}_{i1}^{2}}{\sigma^2} - 1\right]\kappa z_i + (1 - d_i)\ \frac{-\phi(\widetilde{v}_{i2}/\sigma)}{1 - \Phi(\widetilde{v}_{i2}/\hat{\sigma})}\frac{\kappa z_i}{2\widetilde{\sigma}}$$

- As such $LM \xrightarrow{d} \chi^2_{1+\dim(z)}$. The size and power using bootstrap critical values can be estimated from empirical rejection probabilities as before.

8. References

- Box, G. E. P. and D. R. Cox (1964) “An Analysis of Transformations”, *Journal of the Royal Statistical Society*, 26, 211-243.
- Drukker, D. M. (2002) *“Bootstrapping a conditional moments test for normality after tobit* estimation”, The Stata Journal, 2, 125-139
- Moffatt, P. G. (2003) “Hurdle models of loan default” , *School of Economic and Social Studies, University of East Anglia, Norwich, UK*