Description

`exlogistic` fits an exact logistic regression model, which produces more accurate inference in small samples than the standard maximum-likelihood–based logistic regression estimator. It can also better deal with completely determined outcomes. `exlogistic` with the `group(varname)` option conditions on the number of positive outcomes within stratum and is an alternative to the conditional (fixed-effects) logistic regression estimator.

Unlike Stata’s other estimation commands, `exlogistic` must perform hypothesis tests during estimation rather than during postestimation with standard postestimation commands.

Quick start

Exact logistic regression of y on x1, x2, and x3

```stata
exlogistic y x1 x2 x3
```

As above, but condition on values of x3 to save time and memory

```stata
exlogistic y x1 x2, condvars(x3)
```

As above, and allow more memory for computing the conditional distribution of sufficient statistics

```stata
exlogistic y x1 x2, condvars(x3) memory(100m)
```

Using data stored in binomial form with ys successes out of n trials

```stata
exlogistic ys x1 x2 x3, binomial(n)
```

Report coefficients rather than odds ratios

```stata
exlogistic y x1 x2 x3, coef
```

Report conditional scores tests

```stata
exlogistic y x1 x2 x3, test(score)
```

As above, and report joint test for x1 and x2

```stata
exlogistic y x1 x2 x3, test(score) terms(t1=x1 x2)
```

Include strata-specific constant terms for each level of svar for an exact version of conditional logistic regression

```stata
exlogistic y x1 x2 x3, group(svar)
```

Menu

Statistics > Exact statistics > Exact logistic regression
### Syntax

```
exlogistic depvar indepvars [if] [in] [weight] [, options]
```

`depvar` can be specified as a zero or nonzero variable or the number of positive outcomes within each trial. For a zero or nonzero variable, zero indicates failure and nonzero indicates success. To specify `depvar` as the number of positive outcomes, you must also specify `binomial(varname | #)`.

#### options

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<td>`binomial(varname</td>
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#### Options

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#### Reporting

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<th>Description</th>
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by, `statsby`, and `xi` are allowed; see [U] 11.1.10 Prefix commands.

`fweights` are allowed; see [U] 11.1.6 weight.

See [U] 20 Estimation and postestimation commands for more capabilities of estimation commands.

### Options

- `condvars(varlist)` specifies variables whose parameter estimates are not of interest to you. You can save substantial computer time and memory moving such variables from `indepvars` to `condvars()`. Understand that you will get the same results for `x1` and `x3` whether you type

  ```
  . exlogistic y x1 x2 x3 x4
  ```

  or

  ```
  . exlogistic y x1 x3, condvars(x2 x4)
  ```
group(varname) specifies the variable defining the strata, if any. A constant term is assumed for each stratum identified in varname, and the sufficient statistics for indepvars are conditioned on the observed number of successes within each group. This makes the model estimated equivalent to that estimated by clogit, Stata’s conditional logistic regression command (see [R] clogit). group() may not be specified with noconstant or estconstant.

binomial(varname | #) indicates that the data are in binomial form and depvar contains the number of successes. varname contains the number of trials for each observation. If all observations have the same number of trials, you can instead specify the number as an integer. The number of trials must be a positive integer at least as great as the number of successes. If binomial() is not specified, the data are assumed to be Bernoulli, meaning that depvar equaling zero or nonzero records one failure or success.

estconstant estimates the constant term. By default, the models are assumed to have an intercept (constant), but the value of the intercept is not calculated. That is, the conditional distribution of the sufficient statistics for the indepvars is computed given the number of successes in depvar, thus conditioning out the constant term of the model. Use estconstant if you want the estimate of the intercept reported. estconstant may not be specified with group().

noconstant; see [R] Estimation options. noconstant may not be specified with group().

terms(termname = variable ... variable , termname = variable ... variable ...) defines additional terms of the model on which you want exlogistic to perform joint-significance hypothesis tests. By default, exlogistic reports tests individually on each variable in indepvars. For instance, if variables x1 and x3 are in indepvars, and you want to jointly test their significance, specify terms(t1=x1 x3). To also test the joint significance of x2 and x4, specify terms(t1=x1 x3, t2=x2 x4). Each variable can be assigned to only one term.

Joint tests are computed only for the conditional scores tests and the conditional probabilities tests. See test() below.

memory(# [b | k | m | g]) sets a limit on the amount of memory exlogistic can use when computing the conditional distribution of the parameter sufficient statistics. The default is memory(10m), where m stands for megabyte, or 1,048,576 bytes. The following are also available: b stands for byte; k stands for kilobyte, which is equal to 1,024 bytes; and g stands for gigabyte, which is equal to 1,024 megabytes. The minimum setting allowed is 1m and the maximum is 2048m or 2g, but do not attempt to use more memory than is available on your computer. Also see the first technical note under example 4 on counting the conditional distribution.

saving(filename [, replace]) saves the joint conditional distribution to filename. This distribution is conditioned on those variables specified in condvars(). Use replace to replace an existing file with filename. A Stata data file is created containing all the feasible values of the parameter sufficient statistics. The variable names are the same as those in indepvars, in addition to a variable named _f_ containing the feasible value frequencies (sometimes referred to as the condition numbers).

level(#); see [R] Estimation options. The level(#) option will not work on replay because confidence intervals are based on estimator-specific enumerations. To change the confidence level, you must refit the model.
coef reports the estimated coefficients rather than odds ratios (exponentiated coefficients). coef may be specified when the model is fit or upon replay. coef affects only how results are displayed and not how they are estimated.

test(sufficient | score | probability) reports the p-value associated with the observed sufficient statistics, the conditional scores tests, or the conditional probabilities tests, respectively. The default is test(sufficient). If terms() is included in the specification, the conditional scores test and the conditional probabilities test are applied to each term providing conditional inference for several parameters simultaneously. All the statistics are computed at estimation time regardless of which is specified. Each statistic may thus also be displayed postestimation without having to refit the model; see [R] exlogistic postestimation.

mue(varlist) specifies that median unbiased estimates (MUEs) be reported for the variables in varlist. By default, the conditional maximum likelihood estimates (CMLEs) are reported, except for those parameters for which the CMLEs are infinite. Specify mue(_all) if you want MUEs for all the indepvars.

midp instructs exlogistic to use the mid-p-value rule when computing the MUEs, p-values, and confidence intervals. This adjustment is for the discreteness of the distribution and halves the value of the discrete probability of the observed statistic before adding it to the p-value. The mid-p-value rule cannot be applied to MUEs whose corresponding parameter CMLE is infinite.

log and nolog specify whether to display the enumeration log, which shows the progress of computing the conditional distribution of the sufficient statistics. The enumeration log is displayed by default unless you used set iterlog off to suppress it; see set iterlog in [R] set iter.

Remarks and examples

Exact logistic regression is the estimation of the logistic model parameters by using the conditional distribution of the parameter sufficient statistics. The estimates are referred to as the conditional maximum likelihood estimates (CMLEs). This technique was first introduced by Cox and Snell (1989) as an alternative to using maximum likelihood estimation, which can perform poorly for small sample sizes. For stratified data, exact logistic regression is a small-sample alternative to conditional logistic regression. See [R] logit, [R] logistic, and [R] clogit to obtain maximum likelihood estimates (MLEs) for the logistic model and the conditional logistic model. For a comprehensive overview of exact logistic regression, see Mehta and Patel (1995).

Let $Y_i$ denote a Bernoulli random variable where we observe the outcome $Y_i = y_i, i = 1, \ldots, n$. Associated with each independent observation is a $1 \times p$ vector of covariates, $x_i$. We will denote $\pi_i = \Pr(Y_i | x_i)$ and let the logit function model the relationship between $Y_i$ and $x_i$,

$$\log \left( \frac{\pi_i}{1 - \pi_i} \right) = \theta + x_i \beta$$

where the constant term $\theta$ and the $1 \times p$ vector of regression parameters $\beta$ are unknown. The probability of observing $Y_i = y_i, i = 1, \ldots, n$, is

$$\Pr(Y = y) = \prod_{i=1}^{n} \pi_i^{y_i} (1 - \pi_i)^{1-y_i}$$

where $Y = (Y_1, \ldots, Y_n)$ and $y = (y_1, \ldots, y_n)$. The MLEs for $\theta$ and $\beta$ maximize the log of this function.
The sufficient statistics for $\theta$ and $\beta_j$, $j = 1, \ldots, p$, are $M = \sum_{i=1}^{n} Y_i$ and $T_j = \sum_{i=1}^{n} Y_i x_{ij}$, respectively, and we observe $M = m$ and $T_j = t_j$. By default, \texttt{exlogistic} tallies the conditional distribution of $\mathbf{T} = (T_1, \ldots, T_p)$ given $M = m$. This distribution will have a size of $\binom{n}{m}$. (It would have a size of $2^n$ without conditioning on $M = m$.) Denote one of these vectors $\mathbf{T}^{(k)} = (t_1^{(k)}, \ldots, t_p^{(k)})$, $k = 1, \ldots, N$, with combinatorial coefficient (frequency) $c_k$, $\sum_{k=1}^{N} c_k = \binom{n}{m}$. For each independent variable $x_j$, $j = 1, \ldots, p$, we reduce the conditional distribution further by conditioning on all other observed sufficient statistics $T_l = t_l$, $l \neq j$. The conditional probability of observing $T_j = t_j$ has the form

$$\Pr(T_j = t_j \mid T_l = t_l, l \neq j, M = m) = \frac{c \ e^{t_j \beta_j}}{\sum_k c_k e^{t_j^{(k)} \beta_j}}$$

where the sum is over the subset of $\mathbf{T}$ vectors such that $(T_1^{(k)} = t_1, \ldots, T_j^{(k)} = t_j^{(k)}, \ldots, T_p^{(k)} = t_p)$ and $c$ is the combinatorial coefficient associated with the observed $t$. The CMLE for $\beta_j$ maximizes the log of this function.

Specifying nuisance variables in \texttt{condvars()} will reduce the size of the conditional distribution by conditioning on their observed sufficient statistics as well as conditioning on $M = m$. This reduces the amount of memory consumed at the cost of not obtaining regression estimates for those variables specified in \texttt{condvars()}.

Inferences from MLEs rely on asymptotics, and if your sample size is small, these inferences may not be valid. On the other hand, inferences from the CMLEs are exact in the sense that they use the conditional distribution of the sufficient statistics outlined above.

For small datasets, it is common for the dependent variable to be completely determined by the data. Here the MLEs and the CMLEs are unbounded. \texttt{exlogistic} will instead compute the MUE, the regression estimate that places the observed sufficient statistic at the median of the conditional distribution.

### Example 1

One example presented by Mehta and Patel (1995) is data from a prospective study of perinatal infection and human immunodeficiency virus type 1 (HIV-1). We use a variation of this dataset. There was an investigation Hutto et al. (1991) into whether the blood serum levels of glycoproteins CD4 and CD8 measured in infants at 6 months of age might predict their development of HIV infection. The blood serum levels are coded as ordinal values 0, 1, and 2.

```
. use https://www.stata-press.com/data/r16/hiv1
(prospective study of perinatal infection of HIV-1)
. list in 1/5
     +-------------+-----------------+-----------------+
     | hiv | cd4 | cd8 |
     +-------------+-----------------+-----------------+
     1. | 1 | 0 | 0 |
     2. | 0 | 0 | 0 |
     3. | 1 | 0 | 2 |
     4. | 1 | 1 | 0 |
     5. | 0 | 1 | 0 |
```
We first obtain the MLEs from `logistic` so that we can compare the estimates and associated statistics with the CMLEs from `exlogistic`.

```
. logistic hiv cd4 cd8, coef
Logistic regression                         Number of obs =  47
LR chi2(2) =  15.75                      Prob > chi2 = 0.0004
Log likelihood = -20.751687               Pseudo R2 = 0.2751

                      Coef.  Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------  -------------  -------------  --------  -------------  -------------  -------------
    hiv         |   cd4      |    2.541669   .8392231  -3.03   0.002    -4.186517    -.8968223
               |    cd8      |   1.658586   .821113    2.02   0.043     .0492344    3.267938
               |   _cons     |   .5132389   .6809007    0.75    0.451    -.8213019    1.84778

. exlogistic hiv cd4 cd8, coef
```

Enumerating sample-space combinations:
observation 1: enumerations = 2
observation 2: enumerations = 3
(output omitted)
observation 46: enumerations = 601
observation 47: enumerations = 326

```
Exact logistic regression                         Number of obs =  47
Model score =  13.34655                        Pr >= score = 0.0006

                      Coef.  Suff.  2*Pr(Suff.)  [95% Conf. Interval]
-------------  -------------  -------------  -------------  -------------  -------------
    hiv         |   cd4      |   -2.387632     10  0.0004    -4.699633    -.8221807
               |    cd8      |   1.592366     12  0.0528    -.0137905   3.907876
```

`exlogistic` produced a log showing how many records are generated as it processes each observation. The primary purpose of the log is to provide feedback because generating the distribution can be time consuming, but we also see from the last entry that the joint distribution for the sufficient statistics for `cd4` and `cd8` conditioned on the total number of successes has 326 unique values (but a size of $\binom{47}{14} = 341,643,774,795$).

The statistics for `logistic` are based on asymptotics: for a large sample size, each $Z$ statistic will be approximately normally distributed (with a mean of zero and a standard deviation of one) if the associated regression parameter is zero. The question is whether a sample size of 47 is large enough.

On the other hand, the $p$-values computed by `exlogistic` are from the conditional distributions of the sufficient statistics for each parameter given the sufficient statistics for all other parameters. In this sense, these $p$-values are exact. By default, `exlogistic` reports the sufficient statistics for the regression parameters and the probability of observing a more extreme value. These are single-parameter tests for $H_0: \beta_{cd4} = 0$ and $H_0: \beta_{cd8} = 0$ versus the two-sided alternatives. The conditional scores test, located in the coefficient table header, is testing that both $H_0: \beta_{cd4} = 0$ and $H_0: \beta_{cd8} = 0$. We find these $p$-values to be in fair agreement with the Wald and likelihood-ratio tests from `logistic`.

The confidence intervals for `exlogistic` are computed from the exact conditional distributions. The exact confidence intervals are asymmetrical about the estimate and are wider than the normal-based confidence intervals from `logistic`.

Both estimation techniques indicate that the incidence of HIV infection decreases with increasing CD4 blood serum levels and increases with increasing CD8 blood serum levels. The constant term is
missing from the exact logistic coefficient table because we conditioned out its observed sufficient statistic when tallying the joint distribution of the sufficient statistics for the \textit{cd4} and \textit{cd8} parameters.

The \texttt{test()} option provides two other test statistics used in exact logistic: the conditional scores test, \texttt{test(score)}, and the conditional probabilities test, \texttt{test(probability)}. For comparison, we display the individual parameter conditional scores tests.

\begin{verbatim}
. exlogistic, test(score) coef

Exact logistic regression
Number of obs = 47
Model score = 13.34655
Pr >= score = 0.0006

<table>
<thead>
<tr>
<th>hiv</th>
<th>Coef.</th>
<th>Score</th>
<th>Pr&gt;=Score</th>
<th>[95% Conf. Interval]</th>
</tr>
</thead>
<tbody>
<tr>
<td>cd4</td>
<td>-2.387632</td>
<td>12.88022</td>
<td>0.0003</td>
<td>-4.699633</td>
</tr>
<tr>
<td>cd8</td>
<td>1.592366</td>
<td>4.604816</td>
<td>0.0410</td>
<td>-0.0137905</td>
</tr>
</tbody>
</table>
\end{verbatim}

For the probabilities test, the probability statistic is computed from (1) in \textit{Methods and formulas} with $\beta = 0$. For this example, the $p$-value for the probabilities tests matches the scores tests so they are not displayed here.

\section*{Technical note}

Typically, the value of $\theta$, the constant term, is of little interest, as well as perhaps some of the parameters in $\beta$, but we need to include all parameters in the model to correctly specify it. By conditioning out the nuisance parameters, we can reduce the size of the joint conditional distribution that is used to estimate the regression parameters of interest. The \texttt{condvars()} option allows you to specify a \texttt{varlist} of nuisance variables. By default, \texttt{exlogistic} conditions on the sufficient statistic of $\theta$, which is the number of successes. You can save computation time and computer memory by using the \texttt{condvars()} option because infeasible values of the sufficient statistics associated with the variables in \texttt{condvars()} can be dropped from consideration before all $n$ observations are processed.

Specifying some of your independent variables in \texttt{condvars()} will not change the estimated regression coefficients of the remaining independent variables. For instance, in \textit{example 1}, if we instead type

\begin{verbatim}
. exlogistic hiv cd4, condvars(cd8) coef
\end{verbatim}

the regression coefficient for \textit{cd4} (as well as all associated inference) will be identical.

One reason to have multiple variables in \textit{indepvars} is to make conditional inference of several parameters simultaneously by using the \texttt{terms()} option. If you do not wish to test several parameters simultaneously, it may be more efficient to obtain estimates for individual variables by calling \texttt{exlogistic} multiple times with one variable in \textit{indepvars} and all other variables listed in \texttt{condvars()}. The estimates will be the same as those with all variables in \textit{indepvars}.

\section*{Technical note}

If you fit a \texttt{clogit} (see \texttt{[R] clogit}) model to the HIV data from \textit{example 1}, you will find that the estimates differ from those with \texttt{exlogistic}. (To fit the \texttt{clogit} model, you will have to create a group variable that includes all observations.) The regression estimates will be different because \texttt{clogit} conditions on the constant term only, whereas the estimates from \texttt{exlogistic} condition on the sufficient statistic of the other regression parameter as well as the constant term.
Example 2

The HIV data presented in table IV of Mehta and Patel (1995) are in a binomial form, where the variable hiv contains the HIV cases that tested positive and the variable n contains the number of individuals with the same CD4 and CD8 levels, the binomial number-of-trials parameter. Here \textit{depvar} is hiv, and we use the \texttt{binomial(n)} option to identify the number-of-trials variable.

\begin{verbatim}
. use https://www.stata-press.com/data/r16/hiv_n
(prospective study of perinatal infection of HIV-1; binomial form)
. list
  +-----------------+-----------------+-----------------+-----------------+
<table>
<thead>
<tr>
<th>cd4</th>
<th>cd8</th>
<th>hiv</th>
<th>n</th>
</tr>
</thead>
</table>
  1.   0  2  1  1
  2.   1  2  2  2
  3.   0  0  4  7
  4.   1  1  4 12
  5.   2  2  1  3
  6.   1  0  2  7
  7.   2  0  0  2
  8.   2  1  0 13

Further, the \texttt{cd4} and \texttt{cd8} variables of the \texttt{hiv} dataset are actually factor variables, where each has the ordered levels of \{(0,1,2)\}. Another approach to the analysis is to use indicator variables, and following Mehta and Patel (1995), we used a 0–1 coding scheme that will give us the odds ratio of level 0 versus 2 and level 1 versus 2.

\begin{verbatim}
. generate byte cd4_0 = (cd4==0)
. generate byte cd4_1 = (cd4==1)
. generate byte cd8_0 = (cd8==0)
. generate byte cd8_1 = (cd8==1)
. exlogistic hiv cd4_0 cd4_1 cd8_0 cd8_1, terms(cd4=cd4_0 cd4_1, cd8=cd8_0 cd8_1) binomial(n) test(probability) saving(dist, replace) nolog

note: saving distribution to file dist.dta
note: CML estimate for cd4_0 is +inf; computing MUE
note: CML estimate for cd4_1 is +inf; computing MUE
note: CML estimate for cd8_0 is -inf; computing MUE
note: CML estimate for cd8_1 is -inf; computing MUE

Exact logistic regression
Number of obs = 47
Binomial variable: n
Model prob. = 3.19e-06
Pr <= prob. = 0.0011

<table>
<thead>
<tr>
<th>hiv</th>
<th>Odds Ratio</th>
<th>Prob.</th>
<th>Pr&lt;=Prob.</th>
<th>[95% Conf. Interval]</th>
</tr>
</thead>
<tbody>
<tr>
<td>cd4</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>cd4_0</td>
<td>18.82831*</td>
<td>0.007238</td>
<td>0.0072</td>
<td>1.714079 +Inf</td>
</tr>
<tr>
<td>cd4_1</td>
<td>11.53732*</td>
<td>0.0063701</td>
<td>0.0105</td>
<td>1.575285 +Inf</td>
</tr>
<tr>
<td>cd8</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>cd8_0</td>
<td>0.1056887*</td>
<td>0.0289948</td>
<td>0.0290</td>
<td>0 1.072531</td>
</tr>
<tr>
<td>cd8_1</td>
<td>0.0983388*</td>
<td>0.0241503</td>
<td>0.0242</td>
<td>0 .9837203</td>
</tr>
</tbody>
</table>
\end{verbatim}

\texttt{(*) median unbiased estimates (MUE)}

\begin{verbatim}
. matrix list e(sufficient)
e(sufficient)[1,4]
     cd4_0  cd4_1  cd8_0  cd8_1
r1  5  8  6  4
\end{verbatim}
. display e(n_possible)
1091475

Here we used `terms()` to specify two terms in the model, `cd4` and `cd8`, that make up the `cd4` and `cd8` indicator variables. By doing so, we obtained a conditional probabilities test for `cd4`, simultaneously testing both `cd4_0` and `cd4_1`, and for `cd8`, simultaneously testing both `cd8_0` and `cd8_1`. The p-values for the two terms are 0.0055 and 0.0323, respectively.

This example also illustrates instances where the dependent variable is completely determined by the independent variables and CMLEs are infinite. If we try to obtain MLEs, `logistic` will drop each variable and then terminate with a no-data error, error number 2000.

```
. use https://www.stata-press.com/data/r16/hiv_n, clear
(prospective study of perinatal infection of HIV-1; binomial form)
. generate byte cd4_0 = (cd4==0)
. generate byte cd4_1 = (cd4==1)
. generate byte cd8_0 = (cd8==0)
. generate byte cd8_1 = (cd8==1)
. expand n
(39 observations created)
. logistic hiv cd4_0 cd4_1 cd8_0 cd8_1

```

In example 2, `exlogistic` generated the joint conditional distribution of \(T_{\text{cd4}_0}, T_{\text{cd4}_1}, T_{\text{cd8}_0},\) and \(T_{\text{cd8}_1}\) given \(M = 14\) (the number of individuals that tested positive), and for reference, we listed the observed sufficient statistics that are stored in the matrix `e(sufficient)`. Below, we take that distribution and further condition on \(T_{\text{cd4}_1} = 8, T_{\text{cd8}_0} = 6,\) and \(T_{\text{cd8}_1} = 4\), giving the conditional distribution of \(T_{\text{cd4}_0}\). Here we see that the observed sufficient statistic \(T_{\text{cd4}_0} = 5\) is last in the sorted listing or, equivalently, \(T_{\text{cd4}_0}\) is at the domain boundary of the conditional probability distribution. When this occurs, the conditional probability distribution is monotonically increasing in \(\beta_{\text{cd4}_0}\) and a maximum does not exist.

```
. use dist, clear
. keep if cd4_1==8 & cd8_0==6 & cd8_1==4
(4,139 observations deleted)
. list, sep(0)
```

<table>
<thead>
<tr>
<th><em>f</em></th>
<th>cd4_0</th>
<th>cd4_1</th>
<th>cd8_0</th>
<th>cd8_1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>1668667</td>
<td>0</td>
<td>8</td>
<td>6</td>
</tr>
<tr>
<td>2.</td>
<td>18945542</td>
<td>1</td>
<td>8</td>
<td>6</td>
</tr>
<tr>
<td>3.</td>
<td>55801053</td>
<td>2</td>
<td>8</td>
<td>6</td>
</tr>
<tr>
<td>4.</td>
<td>55867350</td>
<td>3</td>
<td>8</td>
<td>6</td>
</tr>
<tr>
<td>5.</td>
<td>17423175</td>
<td>4</td>
<td>8</td>
<td>6</td>
</tr>
<tr>
<td>6.</td>
<td>1091475</td>
<td>5</td>
<td>8</td>
<td>6</td>
</tr>
</tbody>
</table>
When the CMLEs are infinite, the MUEs are computed (Hirji, Tsiatis, and Mehta 1989). For the cd4_0 estimate, we compute the value $\beta_{\text{cd4}_0}$ such that

$$\Pr(T_{\text{cd4}_0} \geq 5 \mid \beta_{\text{cd4}_0} = \beta_{\text{cd4}_0}, T_{\text{cd4}_1} = 8, T_{\text{cd8}_0} = 6, T_{\text{cd8}_1} = 4, M = 14) = 1/2$$

using (1) in Methods and formulas.

The output is in agreement with example 1: there is an increase in risk of HIV infection for a CD4 blood serum level of 0 relative to a level of 2 and for a level of 1 relative to a level of 2; there is a decrease in risk of HIV infection for a CD8 blood serum level of 0 relative to a level of 2 and for a level of 1 relative to a level of 2.

We also displayed $e(n_{\text{possible}})$. This is the combinatorial coefficient associated with the observed sufficient statistics. The same value is found in the $f$ variable of the conditional distribution dataset listed above. The size of the distribution is $\binom{47}{14} = 341,643,774,795$. This can be verified by summing the $f$ variable of the generated conditional distribution dataset.

```
. use dist, clear
. summarize _f_, meanonly
. di %15.1f r(sum)
 341643774795.0
```

Example 3

One can think of exact logistic regression as a covariate-adjusted exact binomial. To demonstrate this point, we will use `exlogistic` to compute a binomial confidence interval for $m$ successes of $n$ trials, by fitting the constant-only model, and we will compare it with the confidence interval computed by `ci proportions` (see [R] ci). We will use the `saving()` option to retain the dataset containing the feasible values for the constant term sufficient statistic, namely, the number of successes, $m$, given $n$ trials and their associated combinatorial coefficients $\binom{n}{m}$, $m = 0, 1, \ldots, n$.

```
. input y
  1. 1
  2. 0
  3. 1
  4. 0
  5. 1
  6. 1
  7. end
. ci proportions y
```

```bash
Variable | Obs | Proportion | Std. Err. | [95% Conf. Interval]
---------|-----|------------|-----------|---------------------
y        | 6   | .6666667   | .1924501  | .2227781 .9567281
```

```
. exlogistic y, estconstant nolog coef saving(binom)
note: saving distribution to file binom.dta
```

Exact logistic regression

```
Number of obs = 6
```

```
<table>
<thead>
<tr>
<th></th>
<th>Coef.</th>
<th>Suff.</th>
<th>2*Pr(Suff.)</th>
<th>[95% Conf. Interval]</th>
</tr>
</thead>
<tbody>
<tr>
<td>_cons</td>
<td>.6931472</td>
<td>4</td>
<td>0.6875</td>
<td>-1.24955 .3096017</td>
</tr>
</tbody>
</table>
```
We use the postestimation program `estat predict` to transform the estimated constant term and its confidence bounds by using the inverse logit function, `invlogit()` (see [FN] Mathematical functions). The standard error for the estimated probability is computed using the delta method.

```
. estat predict
```

<table>
<thead>
<tr>
<th></th>
<th>Predicted</th>
<th>Std. Err.</th>
<th>[95% Conf. Interval]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability</td>
<td>0.6667</td>
<td>0.1925</td>
<td>0.2228 0.9567</td>
</tr>
</tbody>
</table>

```
use binom, replace
list, sep(0)
```

```
<table>
<thead>
<tr>
<th></th>
<th>f_</th>
<th><em>cons</em></th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2.</td>
<td>6</td>
<td>1</td>
</tr>
<tr>
<td>3.</td>
<td>15</td>
<td>2</td>
</tr>
<tr>
<td>4.</td>
<td>20</td>
<td>3</td>
</tr>
<tr>
<td>5.</td>
<td>15</td>
<td>4</td>
</tr>
<tr>
<td>6.</td>
<td>6</td>
<td>5</td>
</tr>
<tr>
<td>7.</td>
<td>1</td>
<td>6</td>
</tr>
</tbody>
</table>
```

Examining the listing of the generated data, the values contained in the variable `_cons_` are the feasible values of $M$, and the values contained in the variable `_f_` are the binomial coefficients $\begin{pmatrix} 6 \\ m \end{pmatrix}$ with total $\sum_{m=0}^{6} \begin{pmatrix} 6 \\ m \end{pmatrix} = 2^6 = 64$. In the coefficient table, the sufficient statistic for the constant term, labeled `Suff.`, is $m = 4$. This value is located at record 5 of the dataset. Therefore, the two-tailed probability of the sufficient statistic is computed as $0.6875 = \frac{2(15 + 6 + 1)}{64}$.

The constant term is the value of $\theta$ that maximizes the probability of observing $M = 4$; see (1) of Methods and formulas:

$$
\Pr(M = 4 | \theta) = \frac{15e^{4\alpha}}{1 + 6e^{\alpha} + 15e^{2\alpha} + 20e^{3\alpha} + 15e^{4\alpha} + 6e^{5\alpha} + e^{6\alpha}}
$$

The maximum is at the value $\theta = \log 2$, which is demonstrated in the figure below.
The lower and upper confidence bounds are the values of $\theta$ such that $\Pr(M \geq 4|\theta) = 0.025$ and $\Pr(M \leq 4|\theta) = 0.025$, respectively. These probabilities are plotted in the figure below for $\theta \in [-2, 4]$.
Example 4

This example demonstrates the `group()` option, which allows the analysis of stratified data. Here the logistic model is

$$\log\left( \frac{\pi_{ik}}{1 - \pi_{ik}} \right) = \theta_k + x_{ki}\beta$$

where $k$ indexes the $s$ strata, $k = 1, \ldots, s$, and $\theta_k$ is the strata-specific constant term whose sufficient statistic is $M_k = \sum_{i=1}^{n_k} Y_{ki}$.

Mehta and Patel (1995) use a case–control study to demonstrate this model, which is useful in comparing the estimates from `exlogistic` and `clogit`. This study was intended to determine the role of birth complications in people with schizophrenia (Garsd 1988). Siblings from seven families took part in the study, and each individual was classified as normal or schizophrenic. A birth complication index is recorded for each individual that ranges from 0, an uncomplicated birth, to 15, a very complicated birth. Some of the frequencies contained in variable `f` are greater than 1, and these count different births at different times where the individual has the same birth complications index, found in variable `BCindex`.

```
use https://www.stata-press.com/data/r16/schizophrenia, clear
(case-control study on birth complications for people with schizophrenia)
list, sepby(family)
```

<table>
<thead>
<tr>
<th>family</th>
<th>BCindex</th>
<th>schizo</th>
<th>f</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>6</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>7</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>15</td>
<td>1</td>
</tr>
<tr>
<td>8</td>
<td>2</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>9</td>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>3</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>11</td>
<td>3</td>
<td>9</td>
<td>1</td>
</tr>
<tr>
<td>12</td>
<td>3</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>13</td>
<td>4</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>14</td>
<td>4</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>15</td>
<td>5</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>16</td>
<td>5</td>
<td>6</td>
<td>0</td>
</tr>
<tr>
<td>17</td>
<td>5</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>18</td>
<td>6</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>19</td>
<td>6</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>20</td>
<td>6</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>21</td>
<td>7</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>22</td>
<td>7</td>
<td>6</td>
<td>1</td>
</tr>
</tbody>
</table>
. exlogistic schizo BCindex [fw=f], group(family) test(score) coef

Enumerating sample-space combinations:
observation 1: enumerations = 2
observation 2: enumerations = 3
observation 3: enumerations = 4
observation 4: enumerations = 5
observation 5: enumerations = 6
observation 6: enumerations = 7
(output omitted)
observation 21: enumerations = 72
observation 22: enumerations = 40

Exact logistic regression
Number of obs = 29
Group variable: family
Number of groups = 7
Obs per group:
min = 2
avg = 4.1
max = 10
Model score = 6.328033
Pr >= score = 0.0167

<table>
<thead>
<tr>
<th>schizo</th>
<th>Coef.</th>
<th>Score</th>
<th>Pr&gt;=Score</th>
<th>[95% Conf. Interval]</th>
</tr>
</thead>
<tbody>
<tr>
<td>BCindex</td>
<td>.3251178</td>
<td>6.328033</td>
<td>0.0167</td>
<td>.0223423</td>
</tr>
</tbody>
</table>

The asymptotic alternative for this model can be estimated using clogit (equivalently, xtlogit, fe) and is listed below for comparison. We must expand the data because clogit will not accept frequency weights if they are not constant within the groups.

. expand f
(7 observations created)

. clogit schizo BCindex, group(family) nolog
note: multiple positive outcomes within groups encountered.

Conditional (fixed-effects) logistic regression
Number of obs = 29
LR chi2(1) = 5.20
Prob > chi2 = 0.0226
Log likelihood = -6.2819819 Pseudo R2 = 0.2927

| schizo | Coef. | Std. Err. | z | P>|z| | [95% Conf. Interval] |
|--------|-------|-----------|---|-----|----------------------|
| BCindex | .3251178 | .1678981 | 1.94 | 0.053 | -.0039565 | .654192 |

Both techniques compute the same regression estimate for the BCindex, which might not be too surprising because both estimation techniques condition on the total number of successes in each group. The difference lies in the p-values and confidence intervals. The p-value testing $H_0: \beta_{BCindex} = 0$ is approximately 0.0167 for the exact conditional scores test and 0.053 for the asymptotic Wald test. Moreover, the exact confidence interval is asymmetric about the estimate and does not contain zero.

Technical note

The memory(#) option limits the amount of memory that exlogistic will consume when computing the conditional distribution of the parameter sufficient statistics. memory() is independent of the data maximum memory setting (see set max_memory in [D] memory), and it is possible
for exlogistic to exceed the memory limit specified in set max_memory without terminating. By default, a log is provided that displays the number of enumerations (the size of the conditional distribution) after processing each observation. Typically, you will see the number of enumerations increase, and then at some point they will decrease as the multivariate shift algorithm (Hirji, Mehta, and Patel 1987) determines that some of the enumerations cannot achieve the observed sufficient statistics of the conditioning variables. When the algorithm is complete, however, it is necessary to store the conditional distribution of the parameter sufficient statistics as a dataset. It is possible, therefore, to get a memory error when the algorithm has completed if there is not enough memory to store the conditional distribution.

**Technical note**

Computing the conditional distributions and reported statistics requires data sorting and numerical comparisons. If there is at least one single-precision variable specified in the model, exlogistic will make comparisons with a relative precision of $2^{-5}$. Otherwise, a relative precision of $2^{-11}$ is used. Be careful if you use recast to promote a single-precision variable to double precision (see [D] recast). You might try listing the data in full precision (maybe %20.15g; see [D] format) to make sure that this is really what you want. See [D] Data types for information on precision of numeric storage types.

**Stored results**

exlogistic stores the following in e():

**Scalars**

- e(N) number of observations
- e(k_groups) number of groups
- e(n_possible) number of distinct possible outcomes where sum(sufficient) equals observed
  - e(sufficient)
- e(n_trials) binomial number-of-trials parameter
- e(sum_y) sum of depvar
- e(k_indvars) number of independent variables
- e(k_terms) number of model terms
- e(k_condvars) number of conditioning variables
- e(condcons) conditioned on the constant(s) indicator
- e(midp) mid-p-value rule indicator
- e(eps) relative difference tolerance

**Macros**

- e(cmd) exlogistic
- e(cmdline) command as typed
- e(title) title in estimation output
- e(depvar) name of dependent variable
- e(indvars) independent variables
- e(condvars) conditional variables
- e(groupvar) group variable
- e(binomial) binomial number-of-trials variable
- e(terms) term names set in option terms() 
- e(level) confidence level
- e(vtype) weight type
- e(wexp) weight expression
- e(datasignature) the checksum
- e(datasignaturevars) variables used in calculation of checksum
- e(properties) b
- e(estat_cmd) program used to implement estat
- e(marginsnotok) predictions disallowed by margins
Methods and formulas

Methods and formulas are presented under the following headings:

Sufficient statistics
Conditional distribution and CMLE
Median unbiased estimates and exact CI
Conditional hypothesis tests
Sufficient-statistic p-value

Sufficient statistics

Let \{Y_1, Y_2, \ldots, Y_n\} be a set of \(n\) independent Bernoulli random variables, each of which can realize two outcomes, \{0, 1\}. For each \(i = 1, \ldots, n\), we observe \(Y_i = y_i\), and associated with each observation is the covariate row vector of length \(p\), \(x_i = (x_{i1}, \ldots, x_{ip})\). Denote \(\beta = (\beta_1, \ldots, \beta_p)^T\) to be the column vector of regression parameters and \(\theta\) to be the constant. The sufficient statistic for \(\beta_j\) is \(T_j = \sum_{i=1}^n Y_i x_{ij}\), \(j = 1, \ldots, p\), and for \(\theta\) is \(M = \sum_{i=1}^n Y_i\). We observe \(T_j = t_j\), \(t_j = \sum_{i=1}^n y_i x_{ij}\), and \(M = m\), \(m = \sum_{i=1}^n y_i\). The probability of observing \((Y_1 = y_1, Y_2 = y_2, \ldots, Y_n = y_n)\) is

\[
Pr(Y_1 = y_1, \ldots, Y_n = y_n \mid \beta, X) = \frac{\exp(m \theta + t \beta)}{\prod_{i=1}^n \{1 + \exp(\theta + x_i \beta)\}}
\]

where \(t = (t_1, \ldots, t_p)\) and \(X = (x_1^T, \ldots, x_n^T)^T\).

The joint distribution of the sufficient statistics \(T\) is obtained by summing over all possible binary sequences \(Y_1, \ldots, Y_n\) such that \(T = t\) and \(M = m\). This probability function is

\[
Pr(T_1 = t_1, \ldots, T_p = t_p, M = m \mid \beta, X) = \frac{c(t, m) \exp(m \theta + t \beta)}{\prod_{i=1}^n \{1 + \exp(\theta + x_i \beta)\}}
\]

where \(c(t, m)\) is the combinatorial coefficient of \((t, m)\) or the number of distinct binary sequences \(Y_1, \ldots, Y_n\) such that \(T = t\) and \(M = m\) (Cox and Snell 1989).
Conditional distribution and CMLE

Without loss of generality, we will restrict our discussion to computing the CMLE of \( \beta_1 \). If we condition on observing \( M = m \) and \( T_2 = t_2, \ldots, T_p = t_p \), the probability function of \( (T_1 \mid \beta_1, T_2 = t_2, \ldots, T_p = t_p, M = m) \) is

\[
Pr(T_1 = t_1 \mid \beta_1, T_2 = t_2, \ldots, T_p = t_p, M = m) = \frac{c(t, m)e^{t_1\beta_1}}{\sum_{u}c(u, t_2, \ldots, t_p, m)e^{u\beta_1}} \tag{1}
\]

where the sum in the denominator is over all possible values of \( T_1 \) such that \( M = m \) and \( T_2 = t_2, \ldots, T_p = t_p \) and \( c(u, t_2, \ldots, t_p, m) \) is the combinatorial coefficient of \( (u, t_2, \ldots, t_p, m) \) (Cox and Snell 1989). The CMLE for \( \beta_1 \) is the value \( \hat{\beta}_1 \) that maximizes the log of (1). This optimization task is carried out by \( ml \), using the conditional frequency distribution of \( (T_1 \mid T_2 = t_2, \ldots, T_p = t_p, M = m) \) as a dataset. Generating the joint conditional distribution is efficiently computed using the multivariate shift algorithm described by Hirji, Mehta, and Patel (1987).

Difficulties in computing \( \hat{\beta}_1 \) arise if the observed \( (T_1 = t_1, \ldots, T_p = t_p, M = m) \) lies on the boundaries of the distribution of \( (T_1 \mid T_2 = t_2, \ldots, T_p = t_p, M = m) \), where the conditional probability function is monotonically increasing (or decreasing) in \( \beta_1 \). Here the CMLE is plus infinity if it is on the upper boundary, \( Pr(T_1 \leq t_1 \mid T_2 = t_2, \ldots, T_p = t_p, M = m) = 1 \), and is minus infinity if it is on the lower boundary of the distribution, \( Pr(T_1 \geq t_1 \mid T_2 = t_2, \ldots, T_p = t_p, M = m) = 1 \). This concept is demonstrated in example 2. When infinite CMLEs occur, the MUE is computed.

Median unbiased estimates and exact CI

The MUE is computed using the technique outlined by Hirji, Tsiatis, and Mehta (1989). First, we find the values of \( \beta_1^{(u)} \) and \( \beta_1^{(l)} \) such that

\[
Pr(T_1 \leq t_1 \mid \beta_1 = \beta_1^{(u)}, T_2 = t_2, \ldots, T_p = t_p, M = m) = \frac{1}{2}
\]

The MUE is then \( \overline{\beta}_1 = (\beta_1^{(l)} + \beta_1^{(u)}) / 2 \). However, if \( T_1 \) is equal to the minimum of the domain of the conditional distribution, \( \beta^{(l)} \) does not exist and \( \overline{\beta}_1 = \beta^{(u)} \). If \( T_1 \) is equal to the maximum of the domain of the conditional distribution, \( \beta^{(u)} \) does not exist and \( \overline{\beta}_1 = \beta^{(l)} \).

Confidence bounds for \( \beta \) are computed similarly, except that we substitute \( \alpha / 2 \) for \( 1 / 2 \) in (2), where \( 1 - \alpha \) is the confidence level. Here \( \beta_1^{(l)} \) would then be the lower confidence bound and \( \beta_1^{(u)} \) would be the upper confidence bound (see example 3).

Conditional hypothesis tests

To test \( H_0: \beta_1 = 0 \) versus \( H_1: \beta_1 \neq 0 \), we obtain the exact \( p \)-value from \( \sum_{u \in E} f_1(u) - f_1(t_1) / 2 \) if the mid-\( p \)-value rule is used and \( \sum_{u \in E} f_1(u) \) otherwise. Here \( E \) is a critical region, and we define \( f_1(u) = Pr(T_1 = u \mid \beta_1 = 0, T_2 = t_2, \ldots, T_p = t_p, M = m) \) for ease of notation. There are two popular ways to define the critical region: the conditional probabilities test and the conditional scores test (Mehta and Patel 1995). The critical region when using the conditional probabilities test is all values of the sufficient statistic for \( \beta_1 \) that have a probability less than or equal to that of the observed \( t_1 \), \( E_p = \{ u : f_1(u) \leq f_1(t_1) \} \). The critical region of the conditional scores test is defined as all values of the sufficient statistic for \( \beta_1 \) such that its score is greater than or equal to that of \( t_1 \),

\[
E_s = \{ u : (u - \mu_1)^2 / \sigma_1^2 \geq (t_1 - \mu_1)^2 / \sigma_1^2 \}
\]

Here \( \mu_1 \) and \( \sigma_1^2 \) are the mean and variance of \( (T_1 \mid \beta_1 = 0, T_2 = t_2, \ldots, T_p = t_p, M = m) \).
The score statistic is defined as
\[
\left\{ \frac{\partial \ell (\beta)}{\partial \beta} \right\}^2 \left[ -E \left\{ \frac{\partial^2 \ell (\beta)}{\partial \beta^2} \right\} \right]^{-1}
\]
evaluated at \( H_0: \beta = 0 \), where \( \ell \) is the log of (1). The score test simplifies to \((t - E[T|\beta])^2 / \var(T|\beta)\) (Hirji 2006), where the mean and variance are computed from the conditional distribution of the sufficient statistic with \( \beta = 0 \) and \( t \) is the observed sufficient statistic.

**Sufficient-statistic p-value**

The \( p \)-value for testing \( H_0: \beta_1 = 0 \) versus the two-sided alternative when \( (T_1 = t_1|T_2 = t_2, \ldots, T_p = t_p) \) is computed as \( 2 \times \min(p_l, p_u) \), where
\[
p_l = \frac{\sum_{u \leq t_1} c(u, t_2, \ldots, t_p, m)}{\sum_u c(u, t_2, \ldots, t_p, m)}
\]
\[
p_u = \frac{\sum_{u \geq t_1} c(u, t_2, \ldots, t_p, m)}{\sum_u c(u, t_2, \ldots, t_p, m)}
\]
It is the probability of observing a more extreme \( T_1 \).

**References**


**Also see**

[R] *exlogistic postestimation* — Postestimation tools for exlogistic

[R] *binreg* — Generalized linear models: Extensions to the binomial family

[R] *clogit* — Conditional (fixed-effects) logistic regression

[R] *expoisson* — Exact Poisson regression

[R] *logistic* — Logistic regression, reporting odds ratios

[R] *logit* — Logistic regression, reporting coefficients

[U] 20 Estimation and postestimation commands