Title

pca — Principal component analysis

Description
pca and pcamat display the eigenvalues and eigenvectors from the principal component analysis (PCA) eigen decomposition. The eigenvectors are returned in orthonormal form, that is, uncorrelated and normalized.

pca can be used to reduce the number of variables or to learn about the underlying structure of the data. pcamat provides the correlation or covariance matrix directly. For pca, the correlation or covariance matrix is computed from the variables in varlist.

Quick start
Principal component analysis of data
Principal component analysis of v1, v2, v3, and v4
pca v1 v2 v3 v4
As above, but retain only 2 components
pca v1 v2 v3 v4, components(2)
As above, but retain only those components with eigenvalues greater than or equal to 0.5
pca v1 v2 v3 v4, mineigen(.5)
Principal component analysis of covariance matrix instead of correlation matrix
pca v1 v2 v3 v4, covariance

Principal component analysis of a correlation matrix
Principal component analysis of matrix C representing the correlations from 1,000 observations
pcamat C, n(1000)
As above, but retain only 4 components
pcamat C, n(1000) components(4)

Menu
pca
Statistics > Multivariate analysis > Factor and principal component analysis > Principal component analysis (PCA)

pcamat
Statistics > Multivariate analysis > Factor and principal component analysis > PCA of a correlation or covariance matrix
Syntax

Principal component analysis of data

```
pca varlist [if] [in] [weight] [, options]
```

Principal component analysis of a correlation or covariance matrix

```
pcamat matname, n(#) [options pcamat_options]
```

`matname` is a $k \times k$ symmetric matrix or a $k(k+1)/2$ long row or column vector containing the upper or lower triangle of the correlation or covariance matrix.

```
options Description

Model 2
components(#) retain maximum of # principal components; `factors()` is a synonym
mineigen(#) retain eigenvalues larger than #; default is 1e-5
correlation perform PCA of the correlation matrix; the default
covariance perform PCA of the covariance matrix
vce(normal) compute VCE of the eigenvalues and vectors assuming multivariate normality
```

Reporting

```
level(#) set confidence level; default is `level(95)`
blanks(#) display loadings as blanks when |loadings| < #
novce suppress display of SEs even though calculated
*means display summary statistics of variables
```

Advanced

```
tol(#) advanced option; see `Options` for details
ignore advanced option; see `Options` for details
norotated display unrotated results, even if rotated results are available (replay only)
```

* `means` is not allowed with `pcamat`.
* `norotated` is not shown in the dialog box.

```
pcamat_options Description

Model
shape(full) `matname` is a square symmetric matrix; the default
shape(lower) `matname` is a vector with the rowwise lower triangle (with diagonal)
shape(upper) `matname` is a vector with the rowwise upper triangle (with diagonal)
names(namelist) variable names; required if `matname` is triangular
forcepsd modifies `matname` to be positive semidefinite
*n(#) number of observations
sds(matname2) vector with standard deviations of variables
means(matname3) vector with means of variables
```

* `n()` is required for `pcamat`. 
bootstrap, by, collect, jackknife, rolling, statsby, and xi are allowed with pca; see [U] 11.1.10 Prefix commands. However, bootstrap and jackknife results should be interpreted with caution; identification of the pca parameters involves data-dependent restrictions, possibly leading to badly biased and overdispersed estimates (Milan and Whittaker 1995).

Weights are not allowed with the bootstrap prefix; see [R] bootstrap. aweight are not allowed with the jackknife prefix; see [R] jackknife. aweights and fweight are allowed with pca; see [U] 11.1.6 weight.

See [U] 20 Estimation and postestimation commands for more capabilities of estimation commands.

### Options

- **Model 2**
  - **components(#) and mineigen(#)** specify the maximum number of components (eigenvectors or factors) to be retained. components() specifies the number directly, and mineigen() specifies it indirectly, keeping all components with eigenvalues greater than the indicated value. The options can be specified individually, together, or not at all. factors() is a synonym for components().

  - **components(#)** sets the maximum number of components (factors) to be retained. pca and pcamat always display the full set of eigenvalues but display eigenvectors only for retained components. Specifying a number larger than the number of variables in varlist is equivalent to specifying the number of variables in varlist and is the default.

  - **mineigen(#)** sets the minimum value of eigenvalues to be retained. The default is 1e-5 or the value of tol() if specified.

  Specifying components() and mineigen() affects only the number of components to be displayed and stored in e(); it does not enforce the assumption that the other eigenvalues are 0. In particular, the standard errors reported when vce(normal) is specified do not depend on the number of retained components.

- **correlation** and **covariance** specify that principal components be calculated for the correlation matrix and covariance matrix, respectively. The default is correlation. Unlike factor analysis, PCA is not scale invariant; the eigenvalues and eigenvectors of a covariance matrix differ from those of the associated correlation matrix. Usually, a PCA of a covariance matrix is meaningful only if the variables are expressed in the same units.

  For pcamat, do not confuse the type of the matrix to be analyzed with the type of matname. Obviously, if matname is a correlation matrix and the option sds() is not specified, it is not possible to perform a PCA of the covariance matrix.

- **vce(none | normal)** specifies whether standard errors are to be computed for the eigenvalues, the eigenvectors, and the (cumulative) percentage of explained variance (confirmatory PCA). These standard errors are obtained assuming multivariate normality of the data and are valid only for a PCA of a covariance matrix. Be cautious if applying these to correlation matrices.

- **level(#)** specifies the confidence level, as a percentage, for confidence intervals. The default is level(95) or as set by set level; see [U] 20.8 Specifying the width of confidence intervals. level() is allowed only with vce(normal).

  - **blanks(#)** shows blanks for loadings with absolute value smaller than #. This option is ignored when specified with vce(normal).

- **novce** suppresses the display of standard errors, even though they are computed, and displays the PCA results in a matrix/table style. You can specify novce during estimation in combination with vce(normal). More likely, you will want to use novce during replay.
means displays summary statistics of the variables over the estimation sample. This option is not available with pcamat.

tol(#) is an advanced, rarely used option and is available only with vce(normal). An eigenvalue, $ev_i$, is classified as being close to zero if $ev_i < tol \times \max(ev)$. Two eigenvalues, $ev_1$ and $ev_2$, are “close” if $|ev_1 - ev_2| < tol \times \max(ev)$. The default is $tol(1e-5)$. See option ignore below and the technical note later in this entry.

ignore is an advanced, rarely used option and is available only with vce(normal). It continues the computation of standard errors and tests, even if some eigenvalues are suspiciously close to zero or suspiciously close to other eigenvalues, violating crucial assumptions of the asymptotic theory used to estimate standard errors and tests. See the technical note later in this entry.

The following option is available with pca and pcamat but is not shown in the dialog box:

norotated displays the unrotated principal components, even if rotated components are available. This option may be specified only when replaying results.

Options unique to pcamat

shape(shape_arg) specifies the shape (storage mode) for the covariance or correlation matrix matname. The following shapes are supported:

- **full** specifies that the correlation or covariance structure of $k$ variables is stored as a symmetric $k \times k$ matrix. Specifying shape(full) is optional in this case.
- **lower** specifies that the correlation or covariance structure of $k$ variables is stored as a vector with $k(k+1)/2$ elements in rowwise lower-triangular order:

  \[
  C_{11} \ C_{21} \ C_{22} \ C_{31} \ C_{32} \ C_{33} \ \ldots \ \ C_{k1} \ C_{k2} \ \ldots \ C_{kk}
  \]

- **upper** specifies that the correlation or covariance structure of $k$ variables is stored as a vector with $k(k+1)/2$ elements in rowwise upper-triangular order:

  \[
  C_{11} \ C_{12} \ C_{13} \ \ldots \ C_{1k} \ C_{22} \ C_{23} \ \ldots C_{2k} \ \ldots C_{(k-1)k-1} \ C_{(k-1)k} \ C_{kk}
  \]

names(namelist) specifies a list of $k$ different names, which are used to document output and to label estimation results and are used as variable names by predict. By default, pcamat verifies that the row and column names of matname and the column or row names of matname2 and matname3 from the sds() and means() options are in agreement. Using the names() option turns off this check.

forcepsd modifies the matrix matname to be positive semidefinite (psd) and so to be a proper covariance matrix. If matname is not positive semidefinite, it will have negative eigenvalues. By setting negative eigenvalues to 0 and reconstructing, we obtain the least-squares positive-semidefinite approximation to matname. This approximation is a singular covariance matrix.

n(#) is required and specifies the number of observations.

sds(matname2) specifies a $k \times 1$ or $1 \times k$ matrix with the standard deviations of the variables. The row or column names should match the variable names, unless the names() option is specified. sds() may be specified only if matname is a correlation matrix.
means(matname) specifies a $k \times 1$ or $1 \times k$ matrix with the means of the variables. The row or column names should match the variable names, unless the names() option is specified. Specify means() if you have variables in your dataset and want to use predict after pcamat.

Remarks and examples

Principal component analysis (PCA) is commonly thought of as a statistical technique for data reduction. It helps you reduce the number of variables in an analysis by describing a series of uncorrelated linear combinations of the variables that contain most of the variance. In addition to data reduction, the eigenvectors from a PCA are often inspected to learn more about the underlying structure of the data.

PCA originated with the work of Pearson (1901) and Hotelling (1933). For an introduction, see van Belle, Fisher, Heagerty, and Lumley (2004) or Afifi, May, and Clark (2012). More advanced treatments are Mardia, Kent, and Bibby (1979, chap. 8), and Rencher and Christensen (2012, chap. 12). For monograph-sized treatments, including extensive discussions of the relationship between PCA and related approaches, see Jackson (2003) and Jolliffe (2002).

The objective of PCA is to find unit-length ($L'L = I$) linear combinations of the variables with the greatest variance. The first principal component has maximal overall variance. The second principal component has maximal variance among all unit-length linear combinations that are uncorrelated to the first principal component, etc. The last principal component has the smallest variance among all unit-length linear combinations of the variables. All principal components combined contain the same information as the original variables, but the important information is partitioned over the components in a particular way: the components are orthogonal, and earlier components contain more information than later components. PCA thus conceived is just a linear transformation of the data. It does not assume that the data satisfy a specific statistical model, though it does require that the data be interval-level data—otherwise taking linear combinations is meaningless.

PCA is scale dependent. The principal components of a covariance matrix and those of a correlation matrix are different. In applied research, PCA of a covariance matrix is useful only if the variables are expressed in commensurable units.

Technical note

Principal components have several useful properties. Some of these are geometric. Both the principal components and the principal scores are uncorrelated (orthogonal) among each other. The $f$ leading principal components have maximal generalized variance among all $f$ unit-length linear combinations.

It is also possible to interpret PCA as a fixed-effects factor analysis with homoskedastic residuals

$$y_{ij} = a_i'b_j + e_{ij} \quad i = 1, \ldots, n \quad j = 1, \ldots, p$$

where $y_{ij}$ are the elements of the matrix $Y$, $a_i$ (scores) and $b_j$ (loadings) are $f$-vectors of parameters, and $e_{ij}$ are independent homoskedastic residuals. (In factor analysis, the scores $a_i$ are random rather than fixed, and the residuals are allowed to be heteroskedastic in $j$.) It follows that $E(Y)$ is a matrix of rank $f$, with $f$ typically substantially less than $n$ or $p$. Thus we may think of PCA as a regression model with a restricted number but unknown independent variables. We may also say that the expected values of the rows (or columns) of $Y$ are in some unknown $f$-dimensional space.

For more information on these properties and for other characterizations of PCA, see Jackson (2003) and Jolliffe (2002).
Example 1: Principal component analysis of audiometric data

We consider a dataset of audiometric measurements on 100 males, age 39. The measurements are minimal discernible intensities at four different frequencies with the left and right ear (see Jackson 2003, 106). The variable $lft1000$ refers to the left ear at 1,000 Hz.

```
. use https://www.stata-press.com/data/r17/audiometric
(Audiometric measures)
. correlate lft* rght* 
(obs=100)

       | lft500  lft1000  lft2000  lft4000  rght500  rght1000  rght2000
----------|------------------
lft500    | 1.0000
lft1000   | 0.7775 1.0000
lft2000   | 0.4012 0.5366 1.0000
lft4000   | 0.2554 0.2749 0.4250 1.0000
rght500   | 0.6963 0.5515 0.2391 0.1790 1.0000
rght1000  | 0.6416 0.7070 0.4460 0.2632 0.6634 1.0000
rght2000  | 0.2372 0.3597 0.7011 0.3165 0.1589 0.4142 1.0000
rght4000  | 0.2041 0.2169 0.3262 0.7097 0.1321 0.2201 0.3746
rght4000  | 1.0000
```

As you may have expected, measurements on the same ear are more highly correlated than measurements on different ears. Also, measurements on different ears at the same frequency are more highly correlated than at different frequencies. Because the variables are in commensurable units, it would make theoretical sense to analyze the covariance matrix of these variables. However, the variances of the measures differ widely:

```
. summarize lft* rght*, sep(4)

Variable | Obs | Mean | Std. dev. | Min  | Max  
----------|-----|------|-----------|------|------
lft500    | 100 | -2.8 | 6.408643  | -10  | 15   
lft1000   | 100 | -0.5 | 7.571211  | -10  | 20   
lft2000   | 100 |  2.0 | 10.94061  | -10  | 45   
lft4000   | 100 | 21.35| 19.61569  | -10  | 70   
rght500   | 100 | -2.6 | 7.123726  | -10  | 25   
rgh1000   | 100 | -0.7 | 6.396811  | -10  | 20   
rgh2000   | 100 |  1.6 | 9.289942  | -10  | 35   
rgh4000   | 100 | 21.35| 19.33039  | -10  | 75   
```

In an analysis of the covariances, the higher frequency measures would dominate the results. There is no clinical reason for such an effect (see also Jackson [2003]). Therefore, we will analyze the correlation matrix.
pca shows two panels. The first panel lists the eigenvalues of the correlation matrix, ordered from largest to smallest. The corresponding eigenvectors are listed in the second panel. These are the principal components and have unit length; the columnwise sum of the squares of the loadings is 1 ($0.4011^2 + 0.4210^2 + \cdots + 0.2542^2 = 1$).

Remark: Literature and software that treat principal components in combination with factor analysis tend to display principal components normed to the associated eigenvalues rather than to 1. This normalization is available in the postestimation command `estat loadings`; see [MV] pca postestimation.

The eigenvalues add up to the sum of the variances of the variables in the analysis—the “total variance” of the variables. Because we are analyzing a correlation matrix, the variables are standardized to have unit variance, so the total variance is 8. The eigenvalues are the variances of the principal components. The first principal component has variance 3.93, explaining 49% (3.93/8) of the total variance. The second principal component has variance 1.62 or 20% (1.62/8) of the total variance. Principal components are uncorrelated. You may want to verify that; for instance,
As a consequence, we may also say that the first two principal components explain the sum of the variances of the individual components, or $49 + 20 = 69\%$ of the total variance. Had the components been correlated, they would have partly represented the same information, so the information contained in the combination would not have been equal to the sum of the information of the components. All eight principal components combined explain all variance in all variables; therefore, the unexplained variances listed in the second panel are all zero, and $\text{Rho} = 1.00$ as shown above the first panel.

More than 85\% of the variance is contained in the first four principal components. We can list just these components with the option `components(4)`.

```
. pca lft* rght*, components(4)
```

Principal components/correlation

<table>
<thead>
<tr>
<th>Component</th>
<th>Eigenvalue</th>
<th>Difference</th>
<th>Proportion</th>
<th>Cumulative</th>
</tr>
</thead>
<tbody>
<tr>
<td>Comp1</td>
<td>3.92901</td>
<td>2.31068</td>
<td>0.4911</td>
<td>0.4911</td>
</tr>
<tr>
<td>Comp2</td>
<td>1.61832</td>
<td>0.54219</td>
<td>0.2023</td>
<td>0.6934</td>
</tr>
<tr>
<td>Comp3</td>
<td>0.97325</td>
<td>0.50552</td>
<td>0.1219</td>
<td>0.8153</td>
</tr>
<tr>
<td>Comp4</td>
<td>0.46682</td>
<td>0.11838</td>
<td>0.0583</td>
<td>0.8737</td>
</tr>
<tr>
<td>Comp5</td>
<td>0.34009</td>
<td>0.12490</td>
<td>0.0425</td>
<td>0.9162</td>
</tr>
<tr>
<td>Comp6</td>
<td>0.31589</td>
<td>0.11492</td>
<td>0.0395</td>
<td>0.9557</td>
</tr>
<tr>
<td>Comp7</td>
<td>0.20011</td>
<td>0.05596</td>
<td>0.0250</td>
<td>0.9807</td>
</tr>
<tr>
<td>Comp8</td>
<td>0.15447</td>
<td>0.00564</td>
<td>0.0193</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

Principal components (eigenvectors)

```

<table>
<thead>
<tr>
<th>Variable</th>
<th>Comp1</th>
<th>Comp2</th>
<th>Comp3</th>
<th>Comp4</th>
<th>Unexplained</th>
</tr>
</thead>
<tbody>
<tr>
<td>lft500</td>
<td>0.4011</td>
<td>-0.3170</td>
<td>0.1582</td>
<td>-0.3278</td>
<td>.1308</td>
</tr>
<tr>
<td>lft1000</td>
<td>0.4210</td>
<td>-0.2255</td>
<td>-0.0520</td>
<td>-0.4816</td>
<td>.1105</td>
</tr>
<tr>
<td>lft2000</td>
<td>0.3664</td>
<td>0.2386</td>
<td>-0.4703</td>
<td>-0.2824</td>
<td>.1275</td>
</tr>
<tr>
<td>lft4000</td>
<td>0.2809</td>
<td>0.4742</td>
<td>0.4295</td>
<td>-0.1611</td>
<td>.1342</td>
</tr>
<tr>
<td>rght500</td>
<td>0.3433</td>
<td>-0.3860</td>
<td>0.2593</td>
<td>0.4876</td>
<td>.1194</td>
</tr>
<tr>
<td>rght1000</td>
<td>0.4114</td>
<td>-0.2318</td>
<td>-0.0289</td>
<td>0.3723</td>
<td>.1825</td>
</tr>
<tr>
<td>rght2000</td>
<td>0.3115</td>
<td>0.3171</td>
<td>-0.5629</td>
<td>0.3914</td>
<td>.07537</td>
</tr>
<tr>
<td>rght4000</td>
<td>0.2542</td>
<td>0.5135</td>
<td>0.4262</td>
<td>0.1591</td>
<td>.1303</td>
</tr>
</tbody>
</table>
```

The first panel is not affected. The second panel now lists the first four principal components. These four components do not contain all information in the data, and therefore some of the variances in the variables are unaccounted for or unexplained. These equal the sums of squares of the loadings in the deleted components, weighted by the associated eigenvalues. The unexplained variances in all variables are of similar order. The average unexplained variance is equal to the overall unexplained variance of 13\% ($1 - 0.87$).

Look more closely at the principal components. The first component has positive loadings of roughly equal size on all variables. It can be interpreted as overall sensitivity of a person’s ears. The second principal component has positive loadings on the higher frequencies with both ears and negative loadings for the lower frequencies. Thus the second principal component distinguishes sensitivity for higher frequencies versus lower frequencies. The third principal component similarly differentiates sensitivity at medium frequencies from sensitivity at other frequencies. Finally, the fourth principal component has negative loadings on the left ear and positive loadings on the right ear; it differentiates the left and right ear.
We stated earlier that the first principal component had similar loadings on all eight variables. This can be tested if we are willing to assume that the data are multivariate normal distributed. For this case, \texttt{pca} can estimate the standard errors and related statistics. To conserve paper, we request only the results of the first two principal components and specify the option \texttt{vce(normal)}.

\begin{verbatim}
. pca l*r*, comp(2) vce(normal)
(with PCA/correlation, SEs and tests are approximate)
Number of obs = 100
Number of comp. = 2
Trace = 8
Rho = 0.6934
SEs assume multivariate normality
SE(Rho) = 0.0273

table

|                | Coefficient | Std. err. | z      | P>|z| | [95% conf. interval] |
|----------------|-------------|-----------|--------|-----|---------------------|
| **Eigenvalues**|             |           |        |     |                     |
| Comp1          |             |           |        |     |                     |
| lft500         | 0.4010948   | 0.0429963 | 9.33   | 0.000| 0.3168236 0.485366  |
| lft1000        | 0.4209908   | 0.0359372 | 11.71  | 0.000| 0.3505551 0.4914264 |
| lft2000        | 0.3663748   | 0.0463297 | 7.91   | 0.000| 0.2755702 0.4571794 |
| lft4000        | 0.2808559   | 0.0626577 | 4.48   | 0.000| 0.1580491 0.4036628 |
| rght500        | 0.343251    | 0.0528285 | 6.50   | 0.000| 0.2397091 0.446793  |
| rght1000       | 0.4114209   | 0.0374312 | 10.99  | 0.000| 0.3380571 0.4847846 |
| rght2000       | 0.3154834   | 0.0514755 | 5.65   | 0.000| 0.2034612 0.4196354 |
| rght4000       | 0.2524212   | 0.0606808 | 3.85   | 0.000| 0.1247303 0.3837121 |
| Comp2          |             |           |        |     |                     |
| lft500         | -0.3169638  | 0.067871  | -4.67  | 0.000| -0.4499885 -0.1839391|
| lft1000        | -0.2254644  | 0.069887  | -3.37  | 0.001| -0.3567595 -0.0941686|
| lft2000        | 0.2385933   | 0.1079073 | 2.21   | 0.027| 0.0270989 0.4500877 |
| lft4000        | 0.4741545   | 0.0967198 | 4.90   | 0.000| 0.284446  0.6638629 |
| rght500        | -0.3860197  | 0.0803155 | -4.81  | 0.000| -0.5434352 -0.2286042|
| rght1000       | -0.2317725  | 0.0647639 | -3.44  | 0.001| -0.3639994 -0.0955456|
| rght2000       | 0.317059    | 0.1215412 | 2.61   | 0.009| 0.0788427 0.5552752 |
| rght4000       | 0.5135121   | 0.0951842 | 5.39   | 0.000| 0.3269544 0.7000697 |

LR test for independence: chi2(28) = 448.21 Prob > chi2 = 0.0000
LR test for sphericity: chi2(35) = 451.11 Prob > chi2 = 0.0000

Explained variance by components

<table>
<thead>
<tr>
<th>Components</th>
<th>Eigenvalue</th>
<th>Proportion</th>
<th>SE_Prop</th>
<th>Cumulative</th>
<th>SE_Cum</th>
<th>Bias</th>
</tr>
</thead>
<tbody>
<tr>
<td>Comp1</td>
<td>3.929005</td>
<td>0.4911</td>
<td>0.0394</td>
<td>0.4911</td>
<td>0.0394</td>
<td>0.056633</td>
</tr>
<tr>
<td>Comp2</td>
<td>1.618322</td>
<td>0.2023</td>
<td>0.0271</td>
<td>0.6934</td>
<td>0.0273</td>
<td>0.015812</td>
</tr>
<tr>
<td>Comp3</td>
<td>0.9753248</td>
<td>0.1219</td>
<td>0.0178</td>
<td>0.8153</td>
<td>0.0175</td>
<td>-0.14322</td>
</tr>
<tr>
<td>Comp4</td>
<td>0.4667822</td>
<td>0.0583</td>
<td>0.0090</td>
<td>0.8737</td>
<td>0.0127</td>
<td>0.007304</td>
</tr>
<tr>
<td>Comp5</td>
<td>0.34009</td>
<td>0.0425</td>
<td>0.0066</td>
<td>0.9162</td>
<td>0.0092</td>
<td>0.026307</td>
</tr>
<tr>
<td>Comp6</td>
<td>0.3158912</td>
<td>0.0395</td>
<td>0.0062</td>
<td>0.9557</td>
<td>0.0055</td>
<td>-0.057717</td>
</tr>
<tr>
<td>Comp7</td>
<td>0.2001111</td>
<td>0.0250</td>
<td>0.0040</td>
<td>0.9807</td>
<td>0.0031</td>
<td>-0.013961</td>
</tr>
<tr>
<td>Comp8</td>
<td>0.1544736</td>
<td>0.0193</td>
<td>0.0031</td>
<td>1.0000</td>
<td>0.0000</td>
<td>-0.020087</td>
</tr>
</tbody>
</table>
\end{verbatim}

Here \texttt{pca} acts like an estimation command. The output is organized in different equations. The first equation contains the eigenvalues. The second equation named, Comp1, is the first principal component, etc. \texttt{pca} reports, for instance, standard errors of the eigenvalues. Although testing the values of eigenvalues may, up to now, be rare in applied research, interpretation of results should take stability into consideration. It makes little sense to report the first eigenvalue as 3.929 if you see that the standard error is 0.56.
pca has also reported the standard errors of the principal components. It has also estimated the covariances.

```
. estat vce
(output omitted)
```

Showing the large amount of information contained in the VCE matrix is not useful by itself. The fact that it has been estimated, however, enables us to test properties of the principal components. Does it make good sense to talk about the loadings of the first principal component being of the same size? We use `testparm` with two options; see [R] `test`. `eq(Comp1)` specifies that we are testing coefficients for equation `Comp1`, that is, the first principal component. `equal` specifies that instead of testing that the coefficients are zero, we want to test that the coefficients are equal to each other—a more sensible hypothesis because principal components are normalized to 1.

```
. testparm lft* rght*, equal eq(Comp1)
( 1)  - [Comp1]lft500 + [Comp1]lft1000 = 0
( 2)  - [Comp1]lft500 + [Comp1]lft2000 = 0
( 3)  - [Comp1]lft500 + [Comp1]lft4000 = 0
( 4)  - [Comp1]lft500 + [Comp1]rght500 = 0
( 5)  - [Comp1]lft500 + [Comp1]rght1000 = 0
( 6)  - [Comp1]lft500 + [Comp1]rght2000 = 0
( 7)  - [Comp1]lft500 + [Comp1]rght4000 = 0
    chi2( 7) =  7.56
    Prob > chi2 =  0.3729
```

We cannot reject the null hypothesis of equal loadings, so our interpretation of the first component does not seem to conflict with the data.

pca also displays standard errors of the proportions of variance explained by the leading principal components. Again this information is useful primarily to indicate the strength of formulations of results rather than to test hypotheses about these statistics. The information is also useful to compare studies: if in one study the leading two principal components explain 70% of variance, whereas in a replicating study they explain 80%, are these differences significant given the sampling variation?

Because pca is an estimation command just like `regress` or `xtlogit`, you may replay the output by typing just pca. If you have used pca with the `vce(normal)` option, you may use the option `novce` at estimation or during replay to display the standard pca output.

```
. pca, novce
Principal components/correlation
Number of obs    =     100
Number of comp.  =      2
Trace            =      8
Rotation: (unrotated = principal)
Rho              = 0.6934

<table>
<thead>
<tr>
<th>Component</th>
<th>Eigenvalue</th>
<th>Difference</th>
<th>Proportion</th>
<th>Cumulative</th>
</tr>
</thead>
<tbody>
<tr>
<td>Comp1</td>
<td>3.92901</td>
<td>2.31068</td>
<td>0.4911</td>
<td>0.4911</td>
</tr>
<tr>
<td>Comp2</td>
<td>1.61832</td>
<td>0.642997</td>
<td>0.2023</td>
<td>0.6934</td>
</tr>
<tr>
<td>Comp3</td>
<td>0.975325</td>
<td>0.508543</td>
<td>0.1219</td>
<td>0.8153</td>
</tr>
<tr>
<td>Comp4</td>
<td>0.466782</td>
<td>0.126692</td>
<td>0.0583</td>
<td>0.8737</td>
</tr>
<tr>
<td>Comp5</td>
<td>0.34009</td>
<td>0.0241988</td>
<td>0.0425</td>
<td>0.9162</td>
</tr>
<tr>
<td>Comp6</td>
<td>0.315891</td>
<td>0.11578</td>
<td>0.0395</td>
<td>0.9557</td>
</tr>
<tr>
<td>Comp7</td>
<td>0.20011</td>
<td>0.0456375</td>
<td>0.0250</td>
<td>0.9807</td>
</tr>
<tr>
<td>Comp8</td>
<td>0.154474</td>
<td></td>
<td>0.0193</td>
<td>1.0000</td>
</tr>
</tbody>
</table>
```
Principal components (eigenvectors)

<table>
<thead>
<tr>
<th>Variable</th>
<th>Comp1</th>
<th>Comp2</th>
<th>Unexplained</th>
</tr>
</thead>
<tbody>
<tr>
<td>lft500</td>
<td>0.4011</td>
<td>-0.3170</td>
<td>.2053</td>
</tr>
<tr>
<td>lft1000</td>
<td>0.4210</td>
<td>-0.2255</td>
<td>.2214</td>
</tr>
<tr>
<td>lft2000</td>
<td>0.3664</td>
<td>0.2386</td>
<td>.3805</td>
</tr>
<tr>
<td>lft4000</td>
<td>0.2809</td>
<td>0.4742</td>
<td>.3262</td>
</tr>
<tr>
<td>rght500</td>
<td>0.3433</td>
<td>-0.3860</td>
<td>.2959</td>
</tr>
<tr>
<td>rght1000</td>
<td>0.4114</td>
<td>-0.2318</td>
<td>.248</td>
</tr>
<tr>
<td>rght2000</td>
<td>0.3115</td>
<td>0.3171</td>
<td>.456</td>
</tr>
<tr>
<td>rght4000</td>
<td>0.2542</td>
<td>0.5135</td>
<td>.3193</td>
</tr>
</tbody>
</table>

Technical note

Inference on the eigenvalues and eigenvectors of a covariance matrix is based on a series of assumptions:

(A1) The variables are multivariate normal distributed.

(A2) The variance–covariance matrix of the observations has all distinct and strictly positive eigenvalues.

Under assumptions A1 and A2, the eigenvalues and eigenvectors of the sample covariance matrix can be seen as maximum likelihood estimates for the population analogues that are asymptotically (multivariate) normally distributed (Anderson 1963; Jackson 2003). See Tyler (1981) for related results for elliptic distributions. Be cautious when interpreting because the asymptotic variances are rather sensitive to violations of assumption A1 (and A2). Wald tests of hypotheses that are in conflict with assumption A2 (for example, testing that the first and second eigenvalues are the same) produce incorrect p-values.

Because the statistical theory for a PCA of a correlation matrix is much more complicated, pca and pcamat compute standard errors and tests of a correlation matrix as if it were a covariance matrix. This practice is in line with the application of asymptotic theory in Jackson (2003). This will usually lead to some underestimation of standard errors, but we believe that this problem is smaller than the consequences of deviations from normality.

You may conduct tests for multivariate normality using the mvtest normality command (see [MV] mvtest normality):

```
. mvtest normality lft* rght*, stats(all)
```

Test for multivariate normality

<table>
<thead>
<tr>
<th>Test</th>
<th>Value</th>
<th>Chi-squared</th>
<th>Prob &gt; Chi-squared</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mardia mSkewness</td>
<td>14.52785</td>
<td>251.052</td>
<td>0.0000</td>
</tr>
<tr>
<td>Mardia mKurtosis</td>
<td>94.53331</td>
<td>33.003</td>
<td>0.0000</td>
</tr>
<tr>
<td>Henze-Zirkler</td>
<td>1.272529</td>
<td>118.563</td>
<td>0.0000</td>
</tr>
<tr>
<td>Doornik-Hansen</td>
<td>1.272529</td>
<td>95.318</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

These tests cast serious doubt on the multivariate normality of the variables. We advise caution in interpreting the inference results. Time permitting, you may want to turn to bootstrap methods for inference on the principal components and eigenvalues, but you should be aware of some serious identification problems in using the bootstrap here (Milan and Whittaker 1995).
Example 2: Analyzing the covariance instead of the correlation

We remarked before that the principal components of a correlation matrix are generally different from the principal components of a covariance matrix. pca defaults to performing the PCA of the correlation matrix. To obtain a PCA of the covariance matrix, specify the covariance option.

```
pca l* r*, comp(4) covariance
```

Principal components/covariance

<table>
<thead>
<tr>
<th>Number of obs</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of comp.</td>
<td>4</td>
</tr>
<tr>
<td>Trace</td>
<td>1154.5</td>
</tr>
</tbody>
</table>

Rotation: (unrotated = principal)

| Rho         | 0.9396     |

<table>
<thead>
<tr>
<th>Component</th>
<th>Eigenvalue</th>
<th>Difference</th>
<th>Proportion</th>
<th>Cumulative</th>
</tr>
</thead>
<tbody>
<tr>
<td>Comp1</td>
<td>706.795</td>
<td>527.076</td>
<td>0.6122</td>
<td>0.6122</td>
</tr>
<tr>
<td>Comp2</td>
<td>179.719</td>
<td>68.3524</td>
<td>0.1557</td>
<td>0.7679</td>
</tr>
<tr>
<td>Comp3</td>
<td>111.366</td>
<td>24.5162</td>
<td>0.0965</td>
<td>0.8643</td>
</tr>
<tr>
<td>Comp4</td>
<td>86.8501</td>
<td>57.4842</td>
<td>0.0752</td>
<td>0.9396</td>
</tr>
<tr>
<td>Comp5</td>
<td>29.366</td>
<td>9.53428</td>
<td>0.0254</td>
<td>0.9650</td>
</tr>
<tr>
<td>Comp6</td>
<td>19.8317</td>
<td>6.67383</td>
<td>0.0172</td>
<td>0.9822</td>
</tr>
<tr>
<td>Comp7</td>
<td>13.1578</td>
<td>5.74352</td>
<td>0.0114</td>
<td>0.9936</td>
</tr>
<tr>
<td>Comp8</td>
<td>7.41432</td>
<td>.</td>
<td>0.0064</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

Principal components (eigenvectors)

<table>
<thead>
<tr>
<th>Variable</th>
<th>Comp1</th>
<th>Comp2</th>
<th>Comp3</th>
<th>Comp4</th>
<th>Unexplained</th>
</tr>
</thead>
<tbody>
<tr>
<td>lft500</td>
<td>0.0835</td>
<td>0.2936</td>
<td>-0.0105</td>
<td>0.3837</td>
<td>7.85</td>
</tr>
<tr>
<td>lft1000</td>
<td>0.1091</td>
<td>0.3982</td>
<td>0.0111</td>
<td>0.3162</td>
<td>11.71</td>
</tr>
<tr>
<td>lft2000</td>
<td>0.2223</td>
<td>0.5578</td>
<td>0.0558</td>
<td>-0.4474</td>
<td>11.13</td>
</tr>
<tr>
<td>lft4000</td>
<td>0.6782</td>
<td>-0.1163</td>
<td>-0.7116</td>
<td>-0.0728</td>
<td>.4024</td>
</tr>
<tr>
<td>rght500</td>
<td>0.0662</td>
<td>0.2779</td>
<td>-0.0226</td>
<td>0.4951</td>
<td>12.42</td>
</tr>
<tr>
<td>rght1000</td>
<td>0.0891</td>
<td>0.3119</td>
<td>0.0268</td>
<td>0.2758</td>
<td>11.14</td>
</tr>
<tr>
<td>rght2000</td>
<td>0.1707</td>
<td>0.3745</td>
<td>0.2721</td>
<td>-0.4496</td>
<td>14.71</td>
</tr>
<tr>
<td>rght4000</td>
<td>0.6560</td>
<td>-0.3403</td>
<td>0.6441</td>
<td>0.1550</td>
<td>.4087</td>
</tr>
</tbody>
</table>

As expected, the results are less clear. The total variance to be analyzed is 1,154.5; this is the sum of the variances of the eight variables, that is, the trace of the covariance matrix. The leading principal components now account for a larger fraction of the variance; this is often the case with covariance matrices where the variables have widely different variances. The principal components are somewhat harder to interpret; mainly the loadings are no longer of roughly comparable size.

Example 3: PCA directly from a correlation matrix

Sometimes you do not have the original data but have only the correlation or covariance matrix. pcamat performs a PCA for such a matrix. To simplify presentation, we use the data on the left ear.

```
correlate lft*, cov
```

(obs=100)

<table>
<thead>
<tr>
<th></th>
<th>lft500</th>
<th>lft1000</th>
<th>lft2000</th>
<th>lft4000</th>
</tr>
</thead>
<tbody>
<tr>
<td>lft500</td>
<td>41.0707</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>lft1000</td>
<td>37.7273</td>
<td>57.3232</td>
<td></td>
<td></td>
</tr>
<tr>
<td>lft2000</td>
<td>28.1313</td>
<td>44.4444</td>
<td>119.697</td>
<td></td>
</tr>
<tr>
<td>lft4000</td>
<td>32.1010</td>
<td>40.8333</td>
<td>91.2121</td>
<td>384.775</td>
</tr>
</tbody>
</table>
Suppose that we have the covariances of the variables but not the original data. `correlate` stores the covariances in `r(C)`, so we can use that matrix and invoke `pcamat` with the options `n(100)`, specifying the number of observations, and `names()`, providing the variable names.

```
. matrix Cfull = r(C)
. pcamat Cfull, comp(2) n(100) names(lft500 lft1000 lft2000 lft4000)
```

```
Principal components/correlation

Number of obs = 100
Number of comp. = 2
Trace = 4

Rotation: (unrotated = principal)

<table>
<thead>
<tr>
<th>Component</th>
<th>Eigenvalue</th>
<th>Difference</th>
<th>Proportion</th>
<th>Cumulative</th>
</tr>
</thead>
<tbody>
<tr>
<td>Comp1</td>
<td>2.37181</td>
<td>1.47588</td>
<td>0.5930</td>
<td>0.5930</td>
</tr>
<tr>
<td>Comp2</td>
<td>.895925</td>
<td>.366238</td>
<td>0.2240</td>
<td>0.8169</td>
</tr>
<tr>
<td>Comp3</td>
<td>.529687</td>
<td>.327106</td>
<td>0.1324</td>
<td>0.9494</td>
</tr>
<tr>
<td>Comp4</td>
<td>.202581</td>
<td>.</td>
<td>0.0506</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

Principal components (eigenvectors)

<table>
<thead>
<tr>
<th>Variable</th>
<th>Comp1</th>
<th>Comp2</th>
<th>Unexplained</th>
</tr>
</thead>
<tbody>
<tr>
<td>lft500</td>
<td>0.5384</td>
<td>-0.4319</td>
<td>.1453</td>
</tr>
<tr>
<td>lft1000</td>
<td>0.5730</td>
<td>-0.3499</td>
<td>.1116</td>
</tr>
<tr>
<td>lft2000</td>
<td>0.4958</td>
<td>0.2955</td>
<td>.3387</td>
</tr>
<tr>
<td>lft4000</td>
<td>0.3687</td>
<td>0.7770</td>
<td>.1367</td>
</tr>
</tbody>
</table>
```

If we had to type in the covariance matrix, to avoid excess typing `pcamat` allows you to provide the covariance (or correlation) matrix with just the upper or lower triangular elements including the diagonal. (Thus, for correlations, you have to enter the 1s for the diagonal.) For example, we could enter the lower triangle of our covariance matrix row by row up to and including the diagonal as a one-row Stata matrix.

```
. matrix Clow = (41.0707, 37.7273, 57.3232, 28.1313, 44.4444, 119.697, 32.101, 40.8333, 91.2121, 384.775)
```

The matrix `Clow` has one row and 10 columns. To make seeing the structure easier, we prefer to enter these numbers in the following way:

```
. matrix Clow = (41.0707, 37.7273, 57.3232, 28.1313, 44.4444, 119.697, 32.101, 40.8333, 91.2121, 384.775)
```

When using the lower or upper triangle stored in a row or column vector, it is not possible to define the variable names as row or column names of the matrix; the option `names()` is required. Moreover, we have to specify the option `shape(lower)` to inform `pcamat` that the vector contains the lower triangle, not the upper triangle.

```
. pcamat Clow, comp(2) shape(lower) n(100) names(lft500 lft1000 lft2000 lft4000)
```

(output omitted)
**Stored results**

`pca` and `pcamat` without the `vce(normal)` option store the following in `e()`:

**Scalars**
- `e(N)` number of observations
- `e(f)` number of retained components
- `e(rho)` fraction of explained variance
- `e(trace)` trace of `e(C)`
- `e(lndet)` ln of the determinant of `e(C)`
- `e(cond)` condition number of `e(C)`

**Macros**
- `e(cmd)` `pca` (even for `pcamat`)
- `e(cmdline)` command as typed
- `e(Ctype)` correlation or covariance
- `e(wtype)` weight type
- `e(wexp)` weight expression
- `e(title)` title in output
- `e(properties)` nob noV eigen
- `e(rotate_cmd)` program used to implement `rotate`
- `e(estat_cmd)` program used to implement `estat`
- `e(predict)` program used to implement `predict`
- `e(marginsnotok)` predictions disallowed by `margins`

**Matrices**
- `e(C)` \( p \times p \) correlation or covariance matrix
- `e(means)` \( 1 \times p \) matrix of means
- `e(sds)` \( 1 \times p \) matrix of standard deviations
- `e(Ev)` \( 1 \times p \) matrix of eigenvalues (sorted)
- `e(L)` \( p \times f \) matrix of eigenvectors = components
- `e(Psi)` \( 1 \times p \) matrix of unexplained variance

**Functions**
- `e(sample)` marks estimation sample

`pca` and `pcamat` with the `vce(normal)` option store the above, as well as the following:

**Scalars**
- `e(v_rho)` variance of `e(rho)`
- `e(chi2_i)` \( \chi^2 \) statistic for test of independence
- `e(df_i)` degrees of freedom for test of independence
- `e(p_i)` \( p \)-value for test of independence
- `e(chi2_s)` \( \chi^2 \) statistic for test of sphericity
- `e(df_s)` degrees of freedom for test of sphericity
- `e(p_s)` \( p \)-value for test of sphericity
- `e(rank)` rank of `e(V)`

**Macros**
- `e(vce)` multivariate normality
- `e(properties)` b V eigen

**Matrices**
- `e(b)` \( 1 \times p + fp \) coefficient vector (all eigenvalues and retained eigenvectors)
- `e(Ev_bias)` \( 1 \times p \) matrix: bias of eigenvalues
- `e(Ev_stats)` \( p \times 5 \) matrix with statistics on explained variance
- `e(V)` variance–covariance matrix of the estimates `e(b)`
Methods and formulas

Methods and formulas are presented under the following headings:

Notation
Inference on eigenvalues and eigenvectors
More general tests for multivariate normal distributions

Notation

Let $C$ be the $p \times p$ correlation or covariance matrix to be analyzed. The spectral or eigen decomposition of $C$ is

$$C = V \Lambda V' = \sum_{i=1}^{p} \lambda_i v_i v_i'$$

$v_i' v_j = \delta_{ij}$ (that is, orthonormality)

$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_p \geq 0$

The eigenvectors $v_i$ are also known as the principal components. The direction (sign) of principal components is not defined. pca returns principal components signed so that $1' v_i > 0$. In PCA, “total variance” equals $\text{trace}(C) = \sum \lambda_j$.

Inference on eigenvalues and eigenvectors

The asymptotic distribution of the eigenvectors $\hat{v}_i$ and eigenvalues $\hat{\lambda}_i$ of a covariance matrix $S$ for a sample from a multivariate normal distribution $N(\mu, \Sigma)$ was derived by Girshick (1939); for more results, see also Anderson (1963) and Jackson (2003). Higher-order expansions are discussed in Lawley (1956). See Tyler (1981) for related results for elliptic distributions. The theory of the exact distribution is rather complicated (Muirhead 1982, chap. 9) and hard to implement. If we assume that eigenvalues of $\Sigma$ are distinct and strictly positive, the eigenvalues and eigenvectors of $S$ are jointly asymptotically multivariate normal distributed with the following moments (up to order $n^{-3}$):

$$E(\hat{\lambda}_i) = \lambda_i \left\{ 1 + \frac{1}{n} \sum_{j \neq i} \frac{\lambda_j}{\lambda_i - \lambda_j} \right\} + O(n^{-3})$$

$$\text{Var}(\hat{\lambda}_i) = \frac{2\lambda_i^2}{n} \left\{ 1 - \frac{1}{n} \sum_{j \neq i} \left( \frac{\lambda_j}{\lambda_i - \lambda_j} \right)^2 \right\} + O(n^{-3})$$

$$\text{Cov}(\hat{\lambda}_i, \hat{\lambda}_j) = \frac{2}{n^2} \left( \frac{\lambda_i \lambda_j}{\lambda_i - \lambda_j} \right)^2 + O(n^{-3})$$

$$\text{Var}(\hat{v}_i) = \frac{1}{n} \sum_{j \neq i} \frac{\lambda_i \lambda_j}{(\lambda_i - \lambda_j)^2} v_j v_j'$$

$$\text{Cov}(\hat{v}_i, \hat{v}_j) = -\frac{1}{n} \frac{\lambda_i \lambda_j}{(\lambda_i - \lambda_j)^2} v_i v_j'$$

For the asymptotic theory of the cumulative proportion of variance explained, see Kshirsagar (1972, 454).
More general tests for multivariate normal distributions

The likelihood-ratio $\chi^2$ test of independence (Basilevsky 1994, 187) is

$$\chi^2 = -\left(n - \frac{2p + 5}{6}\right) \ln \{\det(C)\}$$

with $p(p - 1)/2$ degrees of freedom.

The likelihood-ratio $\chi^2$ test of sphericity (Basilevsky 1994, 192) is

$$\chi^2 = -\left(n - \frac{2p^2 + p + 2}{6p}\right) \left[ \ln \{\det(\tilde{\Lambda})\} - p \ln \left\{ \frac{\text{trace}(\tilde{\Lambda})}{p} \right\} \right]$$

with $(p + 2)(p - 1)/2$ degrees of freedom and with $\tilde{\Lambda}$ the eigenvalues of the correlation matrix.

References


Also see

[MV] pca postestimation — Postestimation tools for pca and pcamat

[MV] alpha — Compute interitem correlations (covariances) and Cronbach’s alpha

[MV] biplot — Biplots

[MV] canon — Canonical correlations

[MV] factor — Factor analysis

[D] corr2data — Create dataset with specified correlation structure

[R] tetrachoric — Tetrachoric correlations for binary variables

[U] 20 Estimation and postestimation commands