#### LinearProgram() — Linear programming

Description Diagnostics Syntax References Remarks and examples Also see Conformability

## Description

The LinearProgram() class finds the parameter vector that minimizes or maximizes the linear objective function subject to some restrictions. The restrictions may be linear equality constraints, linear inequality constraints, lower bounds, or upper bounds.

# Syntax

Syntax is presented under the following headings:

Step 1: Initialization Step 2: Definition of linear programming problem Step 3: Perform optimization Step 4: Display or obtain results Utility function for use in all steps Definition of q Functions defining the linear programming problem q.setCoefficients() and q.getCoefficients() q.setMaxOrMin() and q.getMaxOrMin() q.setEquality() and q.getEquality() *q.setInequality()* and *q.getInequality()* q.setBounds() and q.getBounds() q.setMaxiter() and q.getMaxiter() q.setTol() and q.getTol() q.setTrace() and q.getTrace() Performing optimization q.optimize() Functions for obtaining results q.parameters() q.value() q.iterations() q.converged() q.errorcode(), q.errortext(), and q.returncode() Utility function q.query()

#### Step 1: Initialization

q = LinearProgram()

# Step 2: Definition of linear programming problem

void	<pre>q.setCoefficients(real rowvector coef)</pre>
void	q.setMaxOrMin(string scalar maxormin)
void	<pre>q.setEquality(real matrix ecmat, real colvector rhs)</pre>
void	q.setInequality(real matrix iemat, real colvector rhs)
void	<pre>q.setBounds(real rowvector lowerbd, real rowvector upperbd)</pre>
void	<pre>q.setMaxiter(real scalar maxiter)</pre>
void	q.setTol(real scalar tol)
void	q.setTrace(string scalar trace)
real rowvector	q.getCoefficients()
string scalar	q.getMaxOrMin()
real matrix	q.getEquality()
real matrix	q.getInequality()
real matrix	q.getBounds()
real scalar	q.getMaxiter()
real scalar	q.getTol()
string scalar	q.getTrace()

## Step 3: Perform optimization

real scalar q.optimize()

## Step 4: Display or obtain results

real rowvector	q.parameters()
real scalar	q.value()
real scalar	q.iterations()
real scalar	q.converged()
real scalar	q.errorcode()
string scalar	q.errortext()
real scalar	q.returncode()

#### Utility function for use in all steps

void q.query()

## Definition of q

A variable of type LinearProgram is called an instance of the LinearProgram() class. q is an instance of LinearProgram(), a vector of instances, or a matrix of instances. If you are working interactively, you can create an instance of LinearProgram() by typing

q = LinearProgram()

For a row vector of *n* LinearProgram() instances, type

q = LinearProgram(n)

For an  $m \times n$  matrix of LinearProgram() instances, type

q = LinearProgram(*m*, *n*)

In a function, you would declare one instance of the LinearProgram() class q as a scalar.

```
void myfunc()
{
    class LinearProgram scalar q
    q = LinearProgram()
    ...
}
```

Within a function, you can declare q as a row vector of n instances by typing

```
void myfunc()
{
    class LinearProgram rowvector q
    q = LinearProgram(n)
    ...
}
```

For an  $m \times n$  matrix of instances, type

```
void myfunc()
{
    class LinearProgram matrix q
    q = LinearProgram(m, n)
    ...
}
```

### Functions defining the linear programming problem

At a minimum, you need to tell the LinearProgram() class about the coefficients of the linear objective function you wish to optimize. Optionally, you may specify whether to minimize or maximize the objective function, any equality constraints, any inequality constraints, any lower bounds, and any upper bounds. You may also specify the maximum number of iterations allowed, the convergence tolerance, and whether or not to print computational details.

Each pair of functions includes a q.set function that specifies a setting and a q.get function that returns the current setting.

#### q.setCoefficients() and q.getCoefficients()

*q*.setCoefficients(*coef*) sets the linear objective function coefficients. The coefficients must be set before optimization.

*q*.getCoefficients() returns the linear objective function coefficients (or an empty vector if not specified).

#### q.setMaxOrMin() and q.getMaxOrMin()

*q*.setMaxOrMin(*maxormin*) sets whether to perform maximization or minimization. *maxormin* may be "max" or "min". The default is maximization ("max").

q.getMaxOrMin() returns "max" or "min" according to the current setting.

#### q.setEquality() and q.getEquality()

The equality constraints for a linear programming problem are in the form of linear system  $\mathbf{A}_{\text{EC}}\mathbf{x} = \mathbf{b}_{\text{EC}}$ , where  $\mathbf{A}_{\text{EC}}$  is the equality-constraints (EC) matrix and  $\mathbf{b}_{\text{EC}}$  is the right-hand-side (RHS) vector.

q.setEquality(ecmat, rhs) sets the EC matrix and the RHS vector.

q.getEquality() returns a matrix containing both the EC matrix and the RHS vector. The RHS vector is the last column of the returned matrix. (An empty matrix is returned if equality constraints were not specified.)

#### q.setInequality() and q.getInequality()

The inequality constraints for a linear programming problem are in the form of linear system  $A_{IE} x \le b_{IE}$ , where  $A_{IE}$  is the inequality-constraints (IE) matrix and  $b_{IE}$  is the RHS vector.

q.setInequality(*iemat*, *rhs*) sets the IE matrix and the RHS vector.

*q*.getInequality() returns a matrix containing both the IE matrix and the RHS vector. The RHS vector is the last column of the returned matrix. (An empty matrix is returned if inequality constraints were not specified.)

#### q.setBounds() and q.getBounds()

The parameters may have lower bounds or upper bounds. By default, the lower bound is  $-\infty$  and the upper bound is  $\infty$ .

q.setBounds(*lowerbd*, *upperbd*) sets the lower and upper bounds. Using a missing value as the lower bound indicates  $-\infty$ , and using a missing value as the upper bound indicates  $\infty$ .

q.getBounds() returns a two-row matrix containing the lower and upper bounds.

#### q.setMaxiter() and q.getMaxiter()

q.setMaxiter(*maxiter*) specifies the maximum number of iterations, which must be an integer greater than 0. The default value of *maxiter* is 16000.

q.getMaxiter() returns the current maximum number of iterations.

#### q.setTol() and q.getTol()

q.setTol(tol) specifies the convergence-criterion tolerance, which must be greater than 0. The default value of tol is 1e-8.

q.getTol() returns the currently specified tolerance.

#### q.setTrace() and q.getTrace()

*q*.setTrace(*trace*) sets whether or not to print out computation details. *trace* may be "on" or "off". The default value is "off".

*q*.getTrace() returns the current trace status.

## Performing optimization

#### q.optimize()

q.optimize() invokes the optimization process and returns the value of the objective function at the optimum.

#### Functions for obtaining results

After performing optimization, the functions below provide results including parameters, the value at the optimum, the number of iterations, whether convergence was achieved, error messages, and return codes.

#### q.parameters()

q.parameters() returns the parameter vector that optimizes the objective function; it returns an empty vector prior to performing the optimization.

#### q.value()

q.value() returns the value of the objective function at the optimum; it returns a missing value prior to performing the optimization.

#### q.iterations()

q.iterations() returns the number of iterations.

#### q.converged()

q.converged() returns 1 if the optimization converged and 0 if not.

#### q.errorcode(), q.errortext(), and q.returncode()

q.errorcode() returns the error code generated during the computation; it returns 0 if no error is found.

*q*.errortext() returns an error message corresponding to the error code generated during the computation; it returns an empty string if no error is found.

q.returncode() returns the Stata return code corresponding to the error code generated during the computation.

The error codes and the corresponding Stata return codes are as follows:

Error	Return	
code	code	Error text
1	430	problem is infeasible
2	430	problem is unbounded
3	430	maximum number of iterations has been reached
4	3499	dimensions of coefficients, constraints, and bounds do not conform
5	111	dimension of the parameters is 0

### **Utility function**

You can obtain a report of all settings and results currently stored in a class LinearProgram() instance.

#### q.query()

q.query() with no return value displays the information stored in the class.

# **Remarks and examples**

Remarks are presented under the following headings:

Introduction Details about the interior-point method Examples

#### Introduction

The LinearProgram() class is a Mata class for linear programming.

LinearProgram() uses Mehrotra's (1992) predictor-corrector primal-dual method to optimize the linear programming problems of the form

$$\begin{split} \min_{\mathbf{x}} \text{ or } \max_{\mathbf{x}} \mathbf{c} \mathbf{x}' \\ \text{ such that } \mathbf{A}_{\text{EC}} \mathbf{x}' = \mathbf{b}_{\text{EC}} \\ \mathbf{A}_{\text{IE}} \mathbf{x}' \leq \mathbf{b}_{\text{IE}} \\ \text{ lowerbd } \leq \mathbf{x} \leq \text{ upperbd} \end{split}$$

where  $\mathbf{cx}'$  is the linear objective function,  $\mathbf{A}_{EC}\mathbf{x}' = \mathbf{b}_{EC}$  specifies equality constraints,  $\mathbf{A}_{IE}\mathbf{x}' \leq \mathbf{b}_{IE}$  specifies inequality constraints, **lowerbd** is the lower bound on  $\mathbf{x}$ , and **upperbd** is the upper bound on  $\mathbf{x}$ .

Mehrotra's (1992) predictor-corrector primal-dual method is much faster than the traditional simplex method for large problems. This method is a version of the interior-point method that is widely used today instead of the older simplex method that was widely used in the past. This speed comes at the cost of some accuracy, but the inaccuracy can be removed in practice by lowering the convergence tolerance. Lowering the convergence tolerance will produce answers that are practically the same as those produced by the simplex method.

#### Details about the interior-point method

The simplex method breaks down for large problems, because it repeatedly checks a list of candidate solutions. When the list gets too large, the solution time becomes infeasible.

Instead of repeatedly checking a list of candidate solutions, the interior-point method solves a series of approximations to the original problem. This approximation-based approach makes it feasible to solve large problems that are not feasibly solved by the simplex algorithm. The interior-point method is also much faster than the simplex method on the medium-sized and large-sized problems that are common in statistical applications of linear programming.

Because the simplex algorithm checks a list of candidate solutions, it finds the exact integer solution, when one exists. In contrast, interior-point algorithms find the exact integer solution plus or minus the small positive amount  $\epsilon$ . Reducing the convergence tolerance reduces  $\epsilon$  to a value that is practically 0. (Technically, it will be on the order of  $10^{-16}$ .)

See example 1 for an illustration.

For an introduction to class programming in Mata, see [M-2] class.

#### **Examples**

To solve a linear programming problem, you first use LinearProgram() to get an instance of the class. At a minimum, you must also use setCoefficients() to specify the linear objective function coefficients. In the examples below, we demonstrate both basic and more advanced use of the LinearProgram() class.

## Example 1: A first example

Consider the following linear programming problem:

$$\begin{aligned} \max_{x_1, x_2} 5x_1 + 3x_2 \\ \text{such that} & -x_1 + 11x_2 = 33 \\ & 0.5x_1 - x_2 \leq -3 \\ & 2x_1 + 14x_2 \leq 60 \\ & 2x_1 + x_2 \leq 14.5 \\ & x_1 - 0.4x_2 \leq 5 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{aligned}$$

This problem can be written in matrix form,

$$\begin{aligned} \max_{\mathbf{x}} \mathbf{c} \mathbf{x}' \\ \text{such that } \mathbf{A}_{\text{EC}} \mathbf{x}' &= \mathbf{b}_{\text{EC}} \\ \mathbf{A}_{\text{IE}} \mathbf{x}' &\leq \mathbf{b}_{\text{IE}} \\ \mathbf{x} &\geq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

where c = (5, 3),  $A_{EC} = (-1, 11)$ ,  $b_{EC} = 33$ ,

$$\mathbf{A}_{\rm IE} = \begin{bmatrix} 0.5 & -1 \\ 2 & 14 \\ 2 & 1 \\ 1 & -0.4 \end{bmatrix} \quad \text{and} \quad \mathbf{b}_{\rm IE} = \begin{bmatrix} -3 \\ 60 \\ 14.5 \\ 5 \end{bmatrix}$$

We first define all the coefficients and constraints in matrix form, respectively:

```
: c = (5, 3)
:
: Aec = (-1, 11)
: bec = 33
:
: Aie = (0.5, -1 \ 2, 14 \ 2, 1 \ 1, -0.4)
: bie = (-3 \ 60 \ 14.5 \ 5)
:
: lowerbd = (0, 0)
: upperbd = (., .)
```

Here we use missing in the upper bound to indicate an infinite upper bound. (A missing value used in the lower bound indicates a minus infinite lower bound.)

We generate q as an instance of the class:

: q = LinearProgram()

Then we initialize the coefficients and constraints:

```
: q.setCoefficients(c)
```

```
: q.setEquality(Aec, bec)
```

```
: q.setInequality(Aie, bie)
```

```
: q.setBounds(lowerbd, upperbd)
```

Now we can solve the problem:

```
: q.optimize()
9.00000001
```

We can display the optimal parameters after the solution is found:



As mentioned above, the exact solution for this problem is (0,3), and the simplex method would find it. Lowering the convergence tolerance produces a solution that is practically (0,3).



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#### Example 2: Display information about the linear programming problem

Each instance of the class contains a lot of information about the problem at hand. You can use the member function q.query() to display this information. Several other member functions display specific pieces of information. This example illustrates how to use these functions.

Consider the following linear programming problem:

$$\begin{split} \min_{x_1, x_2, x_3} & x_1 + x_2 \\ \text{such that } x_1 + x_2 + x_3 = 5 \\ & x_1 - x_2 + 2x_3 = 8 \\ & x_1 \geq 1 \\ & 0 \leq x_2 \leq 2 \\ & x_3 \geq 0 \end{split}$$

We can express the problem in matrix form, as we did with the previous example:

```
: c = (1, 1, 0)
: Aec = (1, 1, 1 \ 1, -1, 2)
: bec = (5 \ 8)
: lowerbd = (1, 0, 0)
: upperbd = (., 2, .)
```

To calculate the problem, we define the class instance q:

```
: q = LinearProgram()
```

We can show all the default values by using q.query() before we perform any initialization or computation.

: q.query()			
Settings for LinearProgram()			
Version:	1.00		
Problem setup:			
Perform maximization of the following proble	em		
Objective function size:	1 x	0	
Equality constraint matrix size:	0 x	0	
Equality constraint right-hand-side size:	0 x	1	
Inequality constraint matrix size:	0 x	0	
Inequality constraint right-hand-side size:	0 x	1	
Lower bound size:	1 x	0	
Upper bound size:	1 x	0	
Trace:	off		
Convergence			
Maximum iterations:	16000		
Tolerance:	1.0000e-08		
Current status			
Objective function value:			
Converged:	no		
Note: The function setCoefficients() has not bee	en called.		

Now we initialize the problem with all the information required. First, we define the coefficients of the objective function:

: q.setCoefficients(c)

Because this is a minimization problem, we use q.setMaxOrMin() to change the setting from the default, "max", to "min".

```
: q.setMaxOrMin("min")
```

We then define the equality constraints and bounds, respectively.

```
: q.setEquality(Aec, bec)
```

: q.setBounds(lowerbd, upperbd)

Our default maximum number of iterations is 16,000; Stata will issue a warning message when the maximum number of iterations has been reached. For illustrative purposes, we set the maximum to 2 here and compute the approximation.

```
: q.setMaxiter(2)
: q.optimize()
Warning: Maximum number of iterations has been reached.
.
```

The solution could not be found using only two iterations. Thus, We see a warning, and a missing value (.) is returned as the value of the objective function.

We switch back to a maximum of 16,000 iterations.

```
: q.setMaxiter(16000)
```

We can set the trace to "on" to see the computation details.

```
: q.setTrace("on")
: q.optimize()
Quadrature trace:
Iteration
                  Current function value Current error estimate
   1
                    5.202919005
                                             3.44331e+00
    2
                    .5872208056
                                             6.08052e-01
                    .3333542428
   3
                                             2.63268e-02
   4
                    .3333333396
                                             1.29229e-06
   5
                    .33333333333
                                             6.46181e-11
  1.3333333333
```

We turn off the trace by typing

```
: q.setTrace("off")
```

Now we can list parameters at the minimum:

```
: q.parameters()

1 2 3

1 1 .333333333 3.6666666667
```

We can also display the value of the objective function at the solution.

```
: q.value()
1.3333333333
```

And, we can display the number of iterations used to find the solution.

: q.iterations() 5

No error was found, so the error code and error message are

```
: q.errorcode()
0
: q.errortext()
```

: q.query() Settings for LinearProgram()		
Version:	1.00	
Problem setup: Perform minimization of the following problem	em	
Objective function size:	1 x	3
Equality constraint matrix size:	2 x	3
Equality constraint right-hand-side size:	2 x	1
Inequality constraint matrix size:	0 x	0
Inequality constraint right-hand-side size:	0 x	1
Lower bound size:	1 x	3
Upper bound size:	1 x	3
Trace:	off	
Convergence		
Maximum iterations:	16000	
Tolerance:	1.0000e-08	
Current status		
Objective function value:	1.33333333	
Converged:	yes	
Iterations:	5	

We can show all the values by using q.query() after the computation:

The following three examples use linear programming to solve statistical estimation problems. For some statistical estimation problems, there are many equivalent ways of specifying the corresponding linear programming problems. In our examples, we use a common way of specifying each problem.

#### Example 3: Quantile regression

Linear programming can be used to fit quantile regression models.

We begin with a brief introduction to writing the quantile-regression (QR) estimation problem as a linear programming problem; see [R] **qreg**, Koenker and Hallock (2001), and Koenker and Bassett (1978) for more details.

Minimizing the sum of squared residuals produces an estimator of the coefficients in a mean regression model. Analogously, minimizing the sum of the check function of the residuals produces an estimator of the coefficients in a QR model. For the QR model, we estimate the coefficients of the  $\tau$ th conditional quantile function ( $\beta_{\tau}$ ) by solving

$${\min}_{\pmb{\beta}_{\tau}}\sum_{i=1}^n c_{\tau}(y_i-\mathbf{x}_i \pmb{\beta}_{\tau}')$$

where  $y_i$  is the *i*th observation of the outcome **y**,  $\mathbf{x}_i$  is the *i*th observation of the vector of covariates **x**, n is the number of observations, and  $c_{\tau}(\cdot)$  is the check function. The check function  $c_{\tau}(r_i)$  of the residual  $r_i = y_i - \mathbf{x}_i \beta'_{\tau}$  is given by

$$c_\tau(r_i) = \{\tau - \mathbf{I}(r_i < 0)\}r_i$$

4

where

$$\mathbf{I}(r_i < 0) = \begin{cases} 1 & \text{if } r_i < 0 \\ 0 & \text{otherwise} \end{cases}$$

This minimization problem for the estimator of the coefficients in the QR model can be written as the following linear programming problem:

$$\begin{split} \min_{\substack{\boldsymbol{\beta}_{\tau}, \mathbf{u}, \mathbf{v}}} \tau \mathbf{1}'_{n} \mathbf{u} + (1 - \tau) \mathbf{1}'_{n} \mathbf{v} \\ \text{such that } \mathbf{y} - \mathbf{X} \boldsymbol{\beta}_{\tau} &= \mathbf{u} - \mathbf{v} \\ \mathbf{u} &\geq \mathbf{0}_{n} \\ \mathbf{v} &\geq \mathbf{0}_{n} \end{split}$$

where  $\mathbf{1}_n$  is a vector of 1s,  $\mathbf{0}_n$  is a vector of 0s, **X** is the matrix of observations of the covariates, **y** is the vector of observations of the outcome, and **u** and **v** are added to the inequality constraint to transform it into an equality (in other words, they are slack variables).

The above problem can be rewritten as

$$\min_{\boldsymbol{\beta}_{\tau}, \mathbf{u}, \mathbf{v}} \mathbf{c} \begin{bmatrix} \boldsymbol{\beta}_{\tau} \\ \mathbf{u} \\ \mathbf{v} \end{bmatrix}$$
such that  $\mathbf{A}_{\mathrm{EC}} \begin{bmatrix} \boldsymbol{\beta}_{\tau} \\ \mathbf{u} \\ \mathbf{v} \end{bmatrix} = \mathbf{y}$ 
 $\mathbf{u} \ge \mathbf{0}_n$ 
 $\mathbf{v} \ge \mathbf{0}_n$ 

where

$$\mathbf{c} = \begin{bmatrix} \mathbf{0}_k \\ \tau \mathbf{1}_n \\ (1 - \tau) \mathbf{1}_n \end{bmatrix}' \text{ and } \mathbf{A}_{\mathrm{EC}} = \begin{bmatrix} \mathbf{X} & \mathbf{I}_n & -\mathbf{I}_n \end{bmatrix}$$

where k is the number of covariates in **x** and  $I_n$  is the identity matrix.

Now let us see an example using linear programming for quantile regression.

We use an extract of the dataset on chief-executive officer (CEO) salaries from Wooldridge (2020). This extract was created and is distributed by the Boston College Economics department (see Wooldridge datasets). In addition to salary (salary), the dataset also contains information on the CEOs' age (age), whether they completed college (college) or graduate school (grad), their years of experience with the company (comten) and as CEOs (ceoten), and the company's current profits as a percentage of sales (profmarg).

```
. use https://www.stata-press.com/data/r19/ceosal2 (CEO salaries)
```

Let's begin by using qreg to perform a quantile regression for the 75th quantile using salary as the dependent variable, and age, college, grad, comten, ceoten, and profmarg as independent variables.

. qreg salary	age d	college	grad comt	ten ceote	en pr	ofmarg,	quantil	e(.75)	
Iteration 1:	WLS s	sum of w	reighted o	deviation	1s =	34405	.44		
Iteration 1.	Sum (	of abs	waightad	deviatio		34538	146		
Iteration 2:	Sum o	of abs.	weighted	deviatio	- 6110 - 9110 =	34103	361		
Iteration 3:	Sum o	of abs.	weighted	deviatio	ons =	33839	987		
Iteration 4:	Sum o	of abs.	weighted	deviatio	ons =	33408	942		
Iteration 5:	Sum o	of abs	weighted	deviatio	ons =	33395	286		
Iteration 6:	Sum o	of abs.	weighted	deviatio	ons =	32994	. 187		
Iteration 7:	Sum o	of abs.	weighted	deviatio	ons =	32604	.049		
Iteration 8:	Sum	of abs.	weighted	deviatio	ons =	32458	.071		
Iteration 9:	Sum	of abs.	weighted	deviatio	ons =	32278	.694		
Iteration 10:	Sum o	of abs.	weighted	deviatio	ons =	32164	.004		
Iteration 11:	Sum o	of abs.	weighted	deviatio	ons =	32140	5.02		
Iteration 12:	Sum o	of abs.	weighted	deviatio	ons =	32098	.911		
Iteration 13:	Sum o	of abs.	weighted	deviatio	ons =	3209	1.43		
Iteration 14:	Sum o	of abs.	weighted	deviatio	ons =	32054	.647		
Iteration 15:	Sum o	of abs.	weighted	deviatio	ons =	32003	.957		
Iteration 16:	Sum o	of abs.	weighted	deviatio	ons =	31837	.324		
Iteration 17:	Sum o	of abs.	weighted	deviatio	ons =	31741	.915		
Iteration 18:	Sum o	of abs.	weighted	deviatio	ons =	31515	.552		
Iteration 19:	Sum o	of abs.	weighted	deviatio	ons =	31492	.239		
Iteration 20:	Sum o	of abs.	weighted	deviatio	ons =	3148	38.9		
Iteration 21:	Sum o	of abs.	weighted	deviatio	ons =	31488	.622		
Iteration 22:	Sum o	of abs.	weighted	deviatio	ons =	31483	.585		
Iteration 23:	Sum o	of abs.	weighted	deviatio	ons =	31407	.582		
Iteration 24:	Sum o	of abs.	weighted	deviatio	ons =	31407	.363		
Iteration 25:	Sum o	of abs.	weighted	deviatio	ons =	31398	.938		
Iteration 26:	Sum o	of abs.	weighted	deviatio	ons =	31398	.217		
Iteration 27:	Sum o	of abs.	weighted	deviatio	ons =	31396	.024		
.75 Quantile m	regres	ssion				Nur	nber of	obs =	177
Raw sum of c	leviat	tions 32	2811.25 (a	about 11:	19)				
Min sum of d	leviat	tions 31	396.02			Pse	eudo R2	=	0.0431
salarv	Coet	fficient	Std. ei	rr.	t	P> t.	[95%	conf.	intervall
j									
age	5	.379105	9.44154	19 0	. 57	0.570	-13.2	5867	24.01688
college	-55	54.7379	419.760	05 -1	.32	0.188	-1383	.352	273.8764
grad	18	36.6717	140.361	15 1.	. 33	0.185	-90.4	0431	463.7478
comten	-6	.826164	6.49748	54 -1	.05	0.295	-19.6	5225	5.999919
ceoten	17	7.84588	10.2376	52 1	.74	0.083	-2.36	3353	38.05511
profmarg	4	4654744	3.79100	0-0	.12	0.902	-7.94	8978	7.018029
_cons	12	298.486	672.618	37 1	.93	0.055	-29.	2744	2626.247

The output table contains the estimated coefficients that we will now obtain by linear programming. For further interpretation of the output, see [R] qreg.

We begin by importing the data into Mata and adding a vector of 1s to covariates for the constant term:

. mata:

Now specify that  $\tau$  should be 0.75.

: tau = 0.75

Then we formulate the problem as a linear programming problem using the formulas at the beginning of this example:

```
: n = rows(X)
: k = cols(X)
: c = (J(1, k, 0), tau * J(1, n, 1), (1 - tau) * J(1, n, 1))
: Aec = (X, I(n), -I(n))
: lowerbd = (J(1, k, .), J(1, 2*n, 0))
: upperbd = J(1, 2*n + k, .)
```

We then generate an instance of the class and save the required coefficients and the constraints to it:

```
: q = LinearProgram()
```

```
: q.setCoefficients(c)
```

```
: q.setEquality(Aec, y)
```

```
: q.setBounds(lowerbd, upperbd)
```

```
: q.setMaxOrMin("min")
```

Now we solve it:

```
: q.optimize()
 31396.02428
: x = q.parameters()
: x[1..k]
                   1
                                   2
                                                   3
                                                                   4
 1
        5.379105375
                       -554.7378912
                                        186.6717198
                                                         -6.82616387
                   5
                                                   7
                                   6
 1
        17.84587946
                       -.4654742937
                                        1298.486114
```

These point estimates are the same as those computed by qreg.

#### Example 4: Data envelopment analysis

Production theory is a fundamental component of economic analysis. When studying production, we base our analysis on the concept of a production function. The production function describes how inputs in the production process are turned into outputs. Before the work of Debreu (1951), Koopmans (1951), and Farrell (1957), it was assumed that the input–output relationship had no inefficiency. It is now common to study and measure deviations from efficient production (called efficiency analysis).

A way to measure production efficiency is data envelopment analysis DEA; see Cooper, Seiford, and Tone [2007]; Färe [1988]; Grosskopf and Knox Lovell [1994]; and Färe and Primont [1995]). DEA makes no assumptions about the functional form of the production function and is therefore robust to misspecification.

Intuitively, we know the total output produced by each firm, given their inputs. We can then measure the results of each firm relative to the most efficient firms in our sample. The most efficient firms, those that produce the most for a given set of inputs, form a frontier, an envelope. All other firms are underneath and are relatively inefficient. What we obtain when using DEA is a measure of relative efficiency, a number between 0 and 1, where inefficiency is any number below 1.

DEA uses the weighted sum of outputs over the weighted sum of inputs to measure efficiency, and it can be written as a linear programming problem. For example, to estimate the efficiency of a firm or, more generally, the efficiency of a unit k, we solve

$$\begin{array}{ll} \max_{\mathbf{u},\mathbf{v}} \; \frac{\mathbf{Y}_k \mathbf{u}'}{\mathbf{X}_k \mathbf{v}'} \\ \text{such that} \; \frac{\mathbf{Y}_j \mathbf{u}'}{\mathbf{X}_j \mathbf{v}'} \leq 1 \qquad j = 1, \dots, n \\ \mathbf{u} \geq \mathbf{0}'_p \\ \mathbf{v} \geq \mathbf{0}'_m \end{array}$$

where  $\mathbf{X}_j$  and  $\mathbf{Y}_j$  are the inputs and outputs for unit *j*, respectively. *n* is the number of units, *m* is the number of inputs, and *p* is the number of outputs. This problem can be rewritten as a linear programming problem:

\_ \_ \_

$$\begin{aligned} \max_{\mathbf{u},\mathbf{v}} \mathbf{Y}_{k}\mathbf{u}' \\ \text{such that } \mathbf{X}_{k}\mathbf{v}' &= 1 \\ \mathbf{Y}\mathbf{u}' - \mathbf{X}\mathbf{v}' &\leq \mathbf{0}_{n} \\ \mathbf{u} &\geq \mathbf{0}'_{p} \\ \mathbf{v} &\geq \mathbf{0}'_{m} \end{aligned}$$

Using our notation,

 $\begin{aligned} \max_{\mathbf{u},\mathbf{v}} \mathbf{c} \begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix}' \\ \text{such that } \mathbf{A}_{\text{EC}} \begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix}' &= \mathbf{b}_{\text{EC}} \\ \mathbf{A}_{\text{IE}} \begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix}' &\leq \mathbf{b}_{\text{IE}} \\ \mathbf{u} &\geq \mathbf{0}'_p \\ \mathbf{v} &\geq \mathbf{0}'_m \end{aligned}$ 

We have

$$\mathbf{c} = \begin{bmatrix} \mathbf{Y}_k \\ \mathbf{0}_m \end{bmatrix} \qquad \mathbf{A}_{\mathrm{EC}} = \begin{bmatrix} \mathbf{0}_p' & \mathbf{X}_k \end{bmatrix} \qquad \mathbf{b}_{\mathrm{EC}} = 1 \qquad \mathbf{A}_{\mathrm{IE}} = \begin{bmatrix} \mathbf{Y} & -\mathbf{X} \end{bmatrix} \qquad \mathbf{b}_{\mathrm{IE}} = \mathbf{0}_n$$

Here  $\mathbf{0}_m, \mathbf{0}_n$ , and  $\mathbf{0}_p$  are vectors of 0s.

As an example, we use a sample of 756 fictional firms producing a manufactured good with capital and labor. The firms are hypothesized to use a constant returns-to-scale technology, but the sizes of the firms differ. For more details, see example 2 in [R] frontier. The inputs for the firms will be capital (lncapital) and labor (lnlabor); the output will be the manufactured good (lnoutput).

. use https://www.stata-press.com/data/r19/frontier1

We begin by importing the data into Mata:

4

Now we write the coefficients and constraints as a linear programming problem using the equations above, and we use id = 1 to indicate the first firm.

```
: id = 1
: c = (Y[id, .], J(1, m, 0))
: Aec = (J(1, p, 0), X[id, .])
: bec = 1
: Aie = (Y, -X)
: bie = J(n, 1, 0)
: lowerbd = J(1, m + p, 0)
: upperbd = J(1, m + p, .)
```

We then generate an instance of the LinearProgram() class and store all the required information.

```
: q = LinearProgram()
: q.setCoefficients(c)
: q.setEquality(Aec, bec)
: q.setInequality(Aie, bie)
: q.setBounds(lowerbd, upperbd)
```

Then we can compute the relative efficiency of unit 1, which we defined above.

: q.optimize() .191803712

Because the optimal value is less than 1, we conclude that the first firm is inefficient. If we want to estimate the efficiencies of other firms, we simply change id to the firm of interest. For example, if we change id to 261, we can get the relative efficiency of firm 261:

```
: q.optimize()
1
```

This shows that firm 261 is efficient.

#### Example 5: Dantzig selector

In a linear model, we model the mean of the outcome  $y_i$  as the linear combination  $\mathbf{x}_i \boldsymbol{\beta}'$ , where  $\mathbf{x}_i$  are the covariates and  $\boldsymbol{\beta}$  are the coefficients. In the standard case, the number of covariates k is small relative to the number of observations n. In a high-dimensional regression, k is large relative to n, but we must assume that many of the coefficients  $\boldsymbol{\beta}$  on  $\mathbf{x}_i$  are 0.

The Dantzig selector (Candes and Tao 2007) estimates which of the coefficients are 0 and produces estimates of the coefficients that are not 0.

In a standard linear model, we estimate  $\beta$  by minimizing the sum of the squared residuals. The first-order conditions for this minimization problem are known as the normal equations, and in matrix form, they are

$$\mathbf{X}'(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}) = \mathbf{0}$$

where  $\mathbf{X}$  is a matrix containing the observations of the covariates and  $\mathbf{y}$  is a vector containing the observations of the outcome.

The Dantzig selector solution to the high-dimensional linear model finds the smallest-in-magnitude coefficients that get close to solving the first-order conditions. In math, the Dantzig selector solves

$$\begin{split} \min_{\pmb{\beta}} & \| \pmb{\beta} \|_1 \\ \text{such that } \| \mathbf{X}'(\mathbf{y} - \mathbf{X} \pmb{\beta}) \|_\infty \leq \lambda \end{split}$$

where  $\lambda$  is a constant that parameterizes "close" to solving the first-order conditions. (The least absolute shrinkage and selector operator is another solution to the high-dimensional linear model; see [LASSO] Lasso intro for more details.)

The Dantzig selector problem can be written as the following linear programming problem:

$$\min_{\beta, \mathbf{u}} \mathbf{1}_{k}^{\prime} \mathbf{u}$$
such that  $\mathbf{X}'(\mathbf{y} - \mathbf{X}\beta) \leq \lambda \mathbf{1}_{k}$ 
 $-\mathbf{X}'(\mathbf{y} - \mathbf{X}\beta) \leq \lambda \mathbf{1}_{k}$ 
 $\beta - \mathbf{u} \leq \mathbf{0}_{k}$ 
 $-\beta - \mathbf{u} \leq \mathbf{0}_{k}$ 

Here **u** is a vector of variables used as the upper bounds of the absolute values of  $\beta$ .

The above version is closely related to the original motivation for the Dantzig selection, but it is not in an easy-to-implement form. A ready-to-implement form for the above problem is

$$\begin{split} \min_{\boldsymbol{\beta}, \mathbf{u}} \, \mathbf{c} \begin{bmatrix} \boldsymbol{\beta} \\ \mathbf{u} \end{bmatrix} \\ \text{such that } \mathbf{A}_{\text{IE}} \begin{bmatrix} \boldsymbol{\beta} \\ \mathbf{u} \end{bmatrix} \leq \mathbf{b}_{\text{IE}} \end{split}$$

where

$$\mathbf{c} = \begin{bmatrix} \mathbf{0}_k \\ \mathbf{1}_k \end{bmatrix}' \quad \mathbf{A}_{\mathrm{IE}} = \begin{bmatrix} -\mathbf{X}'\mathbf{X} & \mathbf{0}_{k \times k} \\ \mathbf{X}'\mathbf{X} & \mathbf{0}_{k \times k} \\ \mathbf{I}_k & -\mathbf{I}_k \\ -\mathbf{I}_k & -\mathbf{I}_k \end{bmatrix} \quad \mathbf{b}_{\mathrm{IE}} = \begin{bmatrix} -\mathbf{X}'\mathbf{y} + \lambda\mathbf{1}_k \\ \mathbf{X}'\mathbf{y} + \lambda\mathbf{1}_k \\ \mathbf{0}_k \\ \mathbf{0}_k \end{bmatrix}$$

In this example, we use an extract of the data used in Sunyer et al. (2017) that models how the attention of school children is affected by pollution. Our sample contains 1,096 students, and we model the relationship between the mean-hit reaction time (htime) and other factors, including daily nitrogen dioxide levels (no2), home socioeconomical vulnerability index (sev\_home), school socioeconomical vulnerability index (sev\_sch), school noise levels (noise\_sch), starting age at school (age\_start\_sch), number of young siblings they live with (youngsibl), and home greenness (ndvi\_mn).

We begin by using the dataset and dropping the observations that contain missing values.

```
. use https://www.stata-press.com/data/r19/breathe_lp
. generate byte touse = 1
. markout touse *
. drop if touse == 0
(18 observations deleted)
```

Next we create local macros to hold the names of the outcome variable and the covariates in the model, including some powers and interactions of the original covariates.

```
. local ccontrols "sev_home sev_sch age_start_sch ndvi_mn youngsibl noise_sch"
. local ofinterest "no2"
. local depvar htime
. local indeps 'ofinterest' 'ccontrols' c.('ccontrols')#c.('ccontrols')
```

Then we expand the names of covariates in the model, remove the collinear variables, and save the names back to the same macro.

```
. _rmcoll 'indeps', expand
. local indeps 'r(varlist)'
```

Now we import the data into Mata, remove the mean from the outcome variable, and standardize the covariates.

```
. mata:

. mata:

: X = st_data(., '''indeps'"', "touse")

: y = st_data(., "'depvar'", "touse")

: ybar = mean(y)

: y = y :- ybar

: Xbar = mean(X)

: X = X :- Xbar

: sd_X = sqrt(mean(X:^2))

: X = X :/ sd_X
```

We need a value for  $\lambda$  to complete the problem. Prior to writing this example, we used the method of cross-validation to find a good value for  $\lambda$ ; see Hastie, Tibshirani, and Friedman (2009) for an introduction to cross-validation. Cross-validation specifies that we should set  $\lambda = 4163.558931$ .

We are now ready to specify and solve the linear programming problem.

```
: tmpA = quadcross(X, X)
: tmpb = quadcross(X, y)
: lambda = 4163.558931
: k = cols(X)
: c = (J(1, k, 0), J(1, k, 1))
: Aec = (-tmpA, J(k, k, 0) \ tmpA, J(k, k, 0) \ I(k), -I(k) \ -I(k), -I(k))
: bec = (-tmpb :+ lambda \ tmpb :+ lambda \ J(k, 1, 0) \ J(k, 1, 0))
```

We now define an instance of the LinearProgram() class and put the required information into that instance q.

```
: q = LinearProgram()
```

- : q.setCoefficients(c)
- : q.setInequality(Aec, bec)
- : q.setMaxOrMin("min")

Now solve the problem.

```
: q.optimize()
8.21319472
```

We put the estimates of  $\beta$  into the Mata vector **b** by typing

```
: x = q.parameters()
: b = x[1..k]
```

Out of the 28 estimated parameters, we only want to see which variables were selected, that is, which variables have estimated coefficients that are greater than 0 in absolute value. To show the variable names that have been selected, we type

:	ind	<pre>lep_list = tokens('"'indeps'"'</pre>	)
:	sel	= (abs(b) :>= 1e-6)	
:	"se sel	elected variables are " .ected variables are	
:	sel	.ect(indep_list, sel)	
		1	2
	1	sev_home	c.sev_home#c.age_start_sch
		3	4
	1	c.sev_sch#c.ndvi_mn	c.ndvi_mn#c.noise_sch

Here we consider a coefficient to be 0 when its absolute value is less than  $10^{-6}$ . Recall from *Details* about the interior-point method above and the discussion in example 1 that the interior-point method will not produce solution values that are exactly 0. Lowering the tolerance will make the coefficients that are practically 0 have values closer to 0, but it will not change which variables are selected.

The results indicate that the mean-hit reaction time is related to the home socioeconomical vulnerability index, the home socioeconomical vulnerability index times starting age at school, the school socioeconomical vulnerability index times home greenness, and home greenness times school noise levels.

# Conformability

LinearProgram(	):	
input:		
		void
output:		11
	result:	$1 \times 1$
LinearProgram(	n):	
input:		
	<i>n</i> :	$1 \times 1$
output:	7.	
	result:	$1 \times n$
LinearProgram(	m,n):	
input:		
	<i>m</i> :	$1 \times 1$
	<i>n</i> :	$1 \times 1$
output:		
	result:	$m \times n$
setCoefficient	s( <i>coef</i> ):	
input:		
	coef:	$1 \times N$
output:		
	result:	void
getCoefficient	s():	
input:		
		void
output:		
	result:	$1 \times N$
<pre>setMaxOrMin(mage)</pre>	uxormin):	
input:		
	object:	$1 \times 1$
output:		
	result:	void
<pre>getMaxOrMin():</pre>		
input:		
-		void
output:		
	result:	$1 \times 1$

<pre>setEquality(ecmat, i</pre>	rhs):	
input:		
	ecmat:	$M0 \times N$
output	rns:	$M0 \times 1$
ouipui.	result:	void
getFquality():		
innut.		
input.		void
output:		
	result:	$(M0+1) \times N$
<pre>setInequality(iemat</pre>	, <i>rhs</i> ):	
input:		
	iemat:	$M1 \times N$
	rhs:	$M1 \times 1$
output:	result:	void
getInequality():		
input:		
1		void
output:		
	result:	$(Ml+1) \times N$
<pre>setBounds(lowerbd, i</pre>	upperbd):	
input:		
	lowerbd:	$1 \times N$
	upperbd:	$1 \times N$
output:	1.	. 1
	result:	void
getBounds():		
input:		
		void
output:	maguit	$2 \times N$
	resuit.	$2 \times 1$ V
<pre>setMaxiter(maxiter):</pre>		
input:	<u>.</u> ,	1 1
output	maxiter:	$1 \times 1$
бигриг.	result:	void
actMovitor().		
geomanteer ().		
три.		void
output:		
1	result:	$1 \times 1$

<pre>setTol(tol):</pre>		
input:		
	tol:	$1 \times 1$
output:	rosult	void
	resuit.	voiu
getIol():		
input:		void
output:		volu
1	result:	$1 \times 1$
<pre>setTrace(tra</pre>	ce):	
input:		
	trace:	$1 \times 1$
output:	vocult.	woid
	resuit.	voiu
getTrace():		
input:		void
output:		voiu
-	result:	$1 \times 1$
<pre>optimize():</pre>		
input:		
		void
output:	rosult	$1 \times 1$
	vesuii.	1 ^ 1
parameters(	):	
input:		void
output:		volu
	result:	$1 \times N$
value():		
input:		
		void
output:	wagult.	$1 \sim 1$
	vesuii.	1 × 1
iterations(	):	
input:		void
output:		,010
*	result:	$1 \times 1$

converged():	
input:	
	void
output:	
result:	$1 \times 1$
errorcode():	
input:	
	void
output:	
result:	$1 \times 1$
errortext():	
input:	
	void
output:	
result:	$1 \times 1$
returncode():	
input:	
	void
output:	
result:	$1 \times 1$
query():	
input:	
	void
output:	
	void

# Diagnostics

LinearProgram(), q.set\*(), q.get\*(), q.parameters(), q.value(), q.iterations(), q.converged(), q.errorcode(), q.errortext(), q.returncode(), and q.query() functions abort with an error message when used incorrectly.

q.optimize() aborts with an error message if it is used incorrectly. If q.optimize() runs into numerical difficulties, it returns a missing value and displays a warning message including some details about the problem encountered.

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## Also see

- [M-2] class Object-oriented programming (classes)
- [M-5] moptimize() Model optimization
- [M-5] optimize() Function optimization
- [LASSO] Lasso intro Introduction to lasso
- [R] frontier Stochastic frontier models
- [R] **qreg** Quantile regression
- [R] regress Linear regression

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