gschurd() — Generalized Schur decomposition

**Description**

`gschurd(A, B, T, R, U, V, w, b)` computes the generalized Schur decomposition of two square, numeric matrices, A and B, and the generalized eigenvalues. The decomposition is returned in the Schur-form matrix, T; the upper-triangular matrix, R; and the orthogonal (unitary) matrices, U and V. The generalized eigenvalues are returned in the complex vectors w and b.

`gschurdfgroupby(A, B, f, T, R, U, V, w, b, m)` computes the generalized Schur decomposition of two square, numeric matrices, A and B, and the generalized eigenvalues, and groups the results according to whether a condition on each generalized eigenvalue is satisfied. f is a pointer to the function that implements the condition on each generalized eigenvalue, as discussed below. The number of generalized eigenvalues for which the condition is true is returned in m.

_gschurd() mirrors gschurd(), the difference being that it returns T in A and R in B.

_gschurdfgroupby() mirrors gschurdfgroupby(), the difference being that it returns T in A and R in B.

_gschurdl() and _gschurdfgroupbyl() are the interfaces into the LAPACK routines used to implement the above functions; see [M-1] LAPACK. Their direct use is not recommended.

**Syntax**

```c
void gschurd(A, B, T, R, U, V, w, b)
void _gschurd(A, B, U, V, w, b)
void gschurdfgroupby(A, B, f, T, R, U, V, w, b, m)
void _gschurdfgroupby(A, B, f, U, V, w, b, m)
```

**Remarks and examples**

Remarks are presented under the following headings:

Generalized Schur decomposition
Grouping the results

**Generalized Schur decomposition**

The generalized Schur decomposition of a pair of square, numeric matrices, A and B, can be written as

\[ U' \times A \times V = T \]
\[ U' \times B \times V = R \]
where $T$ is in Schur form, $R$ is upper triangular, and $U$ and $V$ are orthogonal if $A$ and $B$ are real and are unitary if $A$ or $B$ is complex. The complex vectors $w$ and $b$ contain the generalized eigenvalues.

If $A$ and $B$ are real, $T$ is in real Schur form and $R$ is a real upper-triangular matrix. If $A$ or $B$ is complex, $T$ is in complex Schur form and $R$ is a complex upper-triangular matrix.

In the example below, we define $A$ and $B$, obtain the generalized Schur decomposition, and list $T$ and $R$.

```plaintext
A = (6, 2, 8, -1\-3, -4, -6, 4\0, 8, 4, 1\-8, -7, -3, 5)
B = (8, 0, -8, -1\-6, -2, -6, -1\-7, -6, 2, -6\1, -7, 9, 2)
gschurd(A, B, T=., R=., U=., V=., w=., b=.)
```

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>12.99313938</td>
<td>1.746927947</td>
<td>3.931212285</td>
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<tr>
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</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-5.527285433</td>
</tr>
</tbody>
</table>

```
R
```

<table>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>4.406836593</td>
<td>6.869534063</td>
<td>-1.840892081</td>
<td>1.740906311</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>13.88730687</td>
<td>0</td>
<td>-.6995556735</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>9.409495218</td>
<td>-4.659386723</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>9.453808732</td>
</tr>
</tbody>
</table>

```
w
```

<table>
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<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>12.9931394</td>
<td>.409611804+1.83488354i</td>
<td>.024799819-.111092453i</td>
<td>-5.52728543</td>
</tr>
</tbody>
</table>

```
b
```

<table>
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<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4.406836593</td>
<td>4.145676341</td>
<td>.2509986829</td>
<td>9.453808732</td>
</tr>
</tbody>
</table>

Generalized eigenvalues can be obtained by typing

```plaintext
w:/b
```

<table>
<thead>
<tr>
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<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.94840508</td>
<td>.098804579+.442601735i</td>
<td>.098804579-.442601735i</td>
<td>-.584662287</td>
</tr>
</tbody>
</table>

Grouping the results

gschurdgroupby() reorders the generalized Schur decomposition so that a selected group of generalized eigenvalues appears in the leading block of the pair $w$ and $b$. It also reorders the generalized Schur form $T$, $R$, and orthogonal (unitary) matrices, $U$ and $V$, correspondingly.

We must pass gschurdgroupby() a pointer to a function that implements our criterion. The function must accept two arguments, a complex scalar and a real scalar, so that it can receive a generalized eigenvalue, and it must return the real value 0 to indicate rejection and a nonzero real value to indicate selection.
In the following example, we use gschurdgroupby() to put the finite, real, generalized eigenvalues first. One of the arguments to gschurdgroupby() is a pointer to the function onlyreal() which accepts two arguments, a complex scalar and a real scalar that define a generalized eigenvalue. onlyreal() returns 1 if the generalized eigenvalue is finite and real; it returns zero otherwise.

```plaintext
: real scalar onlyreal(complex scalar w, real scalar b)
  > { if(b==0) return(0)
  >   if(Im(w/b)==0) return(1)
  >   return(0)
  > }
```

```plaintext
: gschurdgroupby(A, B, &onlyreal(), T=., R=., U=., V=., w=., b=., m=.)
```

We obtain

```plaintext
: T

<table>
<thead>
<tr>
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<th>2</th>
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<th>4</th>
</tr>
</thead>
<tbody>
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<tr>
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<td>0</td>
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<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>6.337230739</td>
<td>1.752690714</td>
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</tbody>
</table>

: R

<table>
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<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
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<td>1.720793701</td>
</tr>
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</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>-12.5704981</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>9.652818299</td>
</tr>
</tbody>
</table>

: w

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
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<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>12.9931394</td>
<td>-5.95236607</td>
<td>.36499234+.1.63500766i</td>
<td>.36499234-1.63500766i</td>
</tr>
</tbody>
</table>

: b

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
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<td>4.406836593</td>
<td>10.18086202</td>
<td>3.694083258</td>
<td>3.694083258</td>
</tr>
</tbody>
</table>

: w:/b

<table>
<thead>
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<td>.098804579-.442601735i</td>
</tr>
</tbody>
</table>
```

m contains the number of real, generalized eigenvalues

```plaintext
: m
2
```
Conformability

gschurd\((A, B, T, R, U, V, w, b)\):
\[\text{input:}\]
\[\begin{align*}
A & : \ n \times n \\
B & : \ n \times n
\end{align*}\]
\[\text{output:}\]
\[\begin{align*}
T & : \ n \times n \\
R & : \ n \times n \\
U & : \ n \times n \\
V & : \ n \times n \\
w & : \ 1 \times n \\
b & : \ 1 \times n
\end{align*}\]

\_gschurd\((A, B, U, V, w, b)\):
\[\text{input:}\]
\[\begin{align*}
A & : \ n \times n \\
B & : \ n \times n
\end{align*}\]
\[\text{output:}\]
\[\begin{align*}
A & : \ n \times n \\
B & : \ n \times n \\
U & : \ n \times n \\
V & : \ n \times n \\
w & : \ 1 \times n \\
b & : \ 1 \times n
\end{align*}\]

gschurdbroupby\((A, B, f, T, R, U, V, w, b, m)\):
\[\text{input:}\]
\[\begin{align*}
A & : \ n \times n \\
B & : \ n \times n \\
f & : \ 1 \times 1
\end{align*}\]
\[\text{output:}\]
\[\begin{align*}
T & : \ n \times n \\
R & : \ n \times n \\
U & : \ n \times n \\
V & : \ n \times n \\
w & : \ 1 \times n \\
b & : \ 1 \times n \\
m & : \ 1 \times 1
\end{align*}\]
\texttt{gschurdgroupby}(A, B, f, U, V, w, b, m):

\textbf{input:}
\begin{align*}
A: & \quad n \times n \\
B: & \quad n \times n \\
f: & \quad 1 \times 1
\end{align*}

\textbf{output:}
\begin{align*}
A: & \quad n \times n \\
B: & \quad n \times n \\
U: & \quad n \times n \\
V: & \quad n \times n \\
w: & \quad 1 \times n \\
b: & \quad 1 \times n \\
m: & \quad 1 \times 1
\end{align*}

\textbf{Diagnostics}

\texttt{gschurd()} and \texttt{gschurdgroupby()} abort with error if \(A\) or \(B\) is a view.

\texttt{gschurd()}, \texttt{gschurd()}, \texttt{gschurdgroupby()}, and \texttt{gschurdgroupby()} return missing results if \(A\) or \(B\) contains missing values.

\textbf{Also see}

[M-1] \texttt{LAPACK} — The LAPACK linear-algebra routines

[M-5] \texttt{geigensystem()} — Generalized eigenvectors and eigenvalues

[M-5] \texttt{ghessenbergd()} — Generalized Hessenberg decomposition

[M-4] \texttt{Matrix} — Matrix functions