

## frontier — Stochastic frontier models

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## Description

`frontier` fits stochastic production or cost frontier models; the default is a production frontier model. It provides estimators for the parameters of a linear model with a disturbance that is assumed to be a mixture of two components, which have a strictly nonnegative and symmetric distribution, respectively. `frontier` can fit models in which the nonnegative distribution component (a measurement of inefficiency) is assumed to be from a half-normal, exponential, or truncated-normal distribution. See [Kumbhakar and Lovell \(2000\)](#) for a detailed introduction to frontier analysis.

## Quick start

Cobb–Douglas production frontier model of `lny1` as a function of `lnx1` and `lnx2`

```
frontier lny1 lnx1 lnx2
```

As above, but use exponential instead of half-normal distribution for the inefficiency term

```
frontier lny1 lnx1 lnx2, distribution(exponential)
```

Include `x3` as an explanatory variable in the idiosyncratic error variance function

```
frontier lny1 lnx1 lnx2, vhet(x3)
```

As above, and include `x4` as an explanatory variable in the technical inefficiency variance function

```
frontier lny1 lnx1 lnx2, vhet(x3) uhet(x4)
```

Conditional mean model with the mean modeled as a linear function of `x3`

```
frontier lny1 lnx1 lnx2, distribution(tnormal) cm(x3)
```

Cost frontier model of `y2` as a function of `lnx1` and `lnx2`

```
frontier y2 lnx1 lnx2, distribution(tnormal) cost
```

## Menu

Statistics > Linear models and related > Frontier models

## Syntax

```
frontier depvar [indepvars] [if] [in] [weight] [, options]
```

<i>options</i>	Description
Model	
<code>noconstant</code>	suppress constant term
<code>distribution(hnormal)</code>	half-normal distribution for the inefficiency term
<code>distribution(exponential)</code>	exponential distribution for the inefficiency term
<code>distribution(tnormal)</code>	truncated-normal distribution for the inefficiency term
<code>ufrom(<i>matrix</i>)</code>	specify untransformed log likelihood; only with <code>d(tnormal)</code>
<code>cm(<i>varlist</i> [, <code>noconstant</code>])</code>	fit conditional mean model; only with <code>d(tnormal)</code> ; use <code>noconstant</code> to suppress constant term
Model 2	
<code>constraints(<i>constraints</i>)</code>	apply specified linear constraints
<code>collinear</code>	keep collinear variables
<code>uhet(<i>varlist</i> [, <code>noconstant</code>])</code>	explanatory variables for technical inefficiency variance function; use <code>noconstant</code> to suppress constant term
<code>vhhet(<i>varlist</i> [, <code>noconstant</code>])</code>	explanatory variables for idiosyncratic error variance function; use <code>noconstant</code> to suppress constant term
<code>cost</code>	fit cost frontier model; default is production frontier model
SE	
<code>vce(<i>vcetype</i>)</code>	<i>vcetype</i> may be <code>oim</code> , <code>opg</code> , <code>bootstrap</code> , or <code>jackknife</code>
Reporting	
<code>level(#)</code>	set confidence level; default is <code>level(95)</code>
<code>nocnsreport</code>	do not display constraints
<code>display_options</code>	control columns and column formats, row spacing, line width, display of omitted variables and base and empty cells, and factor-variable labeling
Maximization	
<code>maximize_options</code>	control the maximization process; seldom used
<code>coeflegend</code>	display legend instead of statistics

*indepvars* and *varlist* may contain factor variables; see [U] 11.4.3 Factor variables.

`bootstrap`, `by`, `fp`, `jackknife`, `rolling`, and `statsby` are allowed; see [U] 11.1.10 Prefix commands.

Weights are not allowed with the `bootstrap` prefix; see [R] `bootstrap`.

`fweights`, `iwweights`, and `pweights` are allowed; see [U] 11.1.6 weight.

`coeflegend` does not appear in the dialog box.

See [U] 20 Estimation and postestimation commands for more capabilities of estimation commands.

## Options

### Model

`noconstant`; see [R] [estimation options](#).

`distribution(distname)` specifies the distribution for the inefficiency term as half-normal (`hnormal`), exponential, or truncated-normal (`tnormal`). The default is `hnormal`.

`ufrom(matrix)` specifies a  $1 \times K$  matrix of untransformed starting values when the distribution is truncated-normal (`tnormal`). `frontier` can estimate the parameters of the model by maximizing either the log likelihood or a transformed log likelihood (see [Methods and formulas](#)). `frontier` automatically transforms the starting values before passing them on to the transformed log likelihood. The matrix must have the same number of columns as there are parameters to estimate.

`cm(varlist [, noconstant])` may be used only with `distribution(tnormal)`. Here `frontier` will fit a conditional mean model in which the mean of the truncated-normal distribution is modeled as a linear function of the set of covariates specified in *varlist*. Specifying `noconstant` suppresses the constant in the mean function.

### Model 2

`constraints(constraints)`, `collinear`; see [R] [estimation options](#).

By default, when fitting the truncated-normal model or the conditional mean model, `frontier` maximizes a transformed log likelihood. When constraints are applied, `frontier` will maximize the untransformed log likelihood with constraints defined in the untransformed metric.

`uhet(varlist [, noconstant])` specifies that the technical inefficiency component is heteroskedastic, with the variance function depending on a linear combination of *varlist<sub>u</sub>*. Specifying `noconstant` suppresses the constant term from the variance function. This option may not be specified with `distribution(tnormal)`.

`vhhet(varlist [, noconstant])` specifies that the idiosyncratic error component is heteroskedastic, with the variance function depending on a linear combination of *varlist<sub>v</sub>*. Specifying `noconstant` suppresses the constant term from the variance function. This option may not be specified with `distribution(tnormal)`.

`cost` specifies that `frontier` fit a cost frontier model.

### SE

`vce(vcetype)` specifies the type of standard error reported, which includes types that are derived from asymptotic theory (`oim`, `opg`) and that use bootstrap or jackknife methods (`bootstrap`, `jackknife`); see [R] [vce\\_option](#).

### Reporting

`level(#)`; see [R] [estimation options](#).

`nocnsreport`; see [R] [estimation options](#).

`display_options`: `nocl`, `nopvalues`, `noomitted`, `vsquish`, `noemptycells`, `baselevels`, `allbaselevels`, `nofvlabel`, `fvwrap(#)`, `fvwrapon(style)`, `cformat(%fmt)`, `pformat(%fmt)`, `sformat(%fmt)`, and `nolstretch`; see [R] [estimation options](#).

## Maximization

`maximize_options`: `difficult`, `technique(algorithm_spec)`, `iterate(#)`, `[no]log`, `trace`, `gradient`, `showstep`, `hessian`, `showtolerance`, `tolerance(#)`, `ltolerance(#)`, `nrtolerance(#)`, `nonrntolerance`, and `from(init_specs)`; see [R] [maximize](#). These options are seldom used.

Setting the optimization type to `technique(bhhh)` resets the default `vcetype` to `vce(opg)`.

The following option is available with `frontier` but is not shown in the dialog box:

`coeflegend`; see [R] [estimation options](#).

## Remarks and examples

[stata.com](http://www.stata.com)

Stochastic production frontier models were introduced by [Aigner, Lovell, and Schmidt \(1977\)](#) and [Meeusen and van den Broeck \(1977\)](#). Since then, stochastic frontier models have become a popular subfield in econometrics. [Kumbhakar and Lovell \(2000\)](#) provide a good introduction.

`frontier` fits three stochastic frontier models with distinct parameterizations of the inefficiency term and can fit stochastic production or cost frontier models.

Let's review the nature of the stochastic frontier problem. Suppose that a producer has a production function  $f(\mathbf{z}_i, \beta)$ . In a world without error or inefficiency, the  $i$ th firm would produce

$$q_i = f(\mathbf{z}_i, \beta)$$

Stochastic frontier analysis assumes that each firm potentially produces less than it might due to a degree of inefficiency. Specifically,

$$q_i = f(\mathbf{z}_i, \beta)\xi_i$$

where  $\xi_i$  is the level of efficiency for firm  $i$ ;  $\xi_i$  must be in the interval  $(0, 1]$ . If  $\xi_i = 1$ , the firm is achieving the optimal output with the technology embodied in the production function  $f(\mathbf{z}_i, \beta)$ . When  $\xi_i < 1$ , the firm is not making the most of the inputs  $\mathbf{z}_i$  given the technology embodied in the production function  $f(\mathbf{z}_i, \beta)$ . Because the output is assumed to be strictly positive (that is,  $q_i > 0$ ), the degree of technical efficiency is assumed to be strictly positive (that is,  $\xi_i > 0$ ).

Output is also assumed to be subject to random shocks, implying that

$$q_i = f(\mathbf{z}_i, \beta)\xi_i \exp(v_i)$$

Taking the natural log of both sides yields

$$\ln(q_i) = \ln\{f(\mathbf{z}_i, \beta)\} + \ln(\xi_i) + v_i$$

Assuming that there are  $k$  inputs and that the production function is linear in logs, defining  $u_i = -\ln(\xi_i)$  yields

$$\ln(q_i) = \beta_0 + \sum_{j=1}^k \beta_j \ln(z_{ji}) + v_i - u_i \quad (1)$$

Because  $u_i$  is subtracted from  $\ln(q_i)$ , restricting  $u_i \geq 0$  implies that  $0 < \xi_i \leq 1$ , as specified above.

Kumbhakar and Lovell (2000) provide a detailed version of the above derivation, and they show that performing an analogous derivation in the dual cost function problem allows us to specify the problem as

$$\ln(c_i) = \beta_0 + \beta_q \ln(q_i) + \sum_{j=1}^k \beta_j \ln(p_{ji}) + v_i + u_i \quad (2)$$

where  $q_i$  is output,  $z_{ji}$  are input quantities,  $c_i$  is cost, and the  $p_{ji}$  are input prices.

Intuitively, the inefficiency effect is required to lower output or raise expenditure, depending on the specification.

## □ Technical note

The model that `frontier` actually fits is of the form

$$y_i = \beta_0 + \sum_{j=1}^k \beta_j x_{ji} + v_i - s u_i$$

where

$$s = \begin{cases} 1, & \text{for production functions} \\ -1, & \text{for cost functions} \end{cases}$$

so, in the context of the discussion above,  $y_i = \ln(q_i)$ , and  $x_{ji} = \ln(z_{ji})$  for a production function; and for a cost function,  $y_i = \ln(c_i)$ , and the  $x_{ji}$  are the  $\ln(p_{ji})$  and  $\ln(q_i)$ . You must take the natural logarithm of the data before fitting a stochastic frontier production or cost model. `frontier` performs no transformations on the data. □

Different specifications of the  $u_i$  and the  $v_i$  terms give rise to distinct models. `frontier` provides estimators for the parameters of three basic models in which the idiosyncratic component,  $v_i$ , is assumed to be independently  $N(0, \sigma_v)$  distributed over the observations. The basic models differ in their specification of the inefficiency term,  $u_i$ , as follows:

**exponential**: the  $u_i$  are independently exponentially distributed with variance  $\sigma_u^2$

**hnormal**: the  $u_i$  are independently half-normally  $N^+(0, \sigma_u^2)$  distributed

**tnormal**: the  $u_i$  are independently  $N^+(\mu, \sigma_u^2)$  distributed with truncation point at 0

For half-normal or exponential distributions, `frontier` can fit models with heteroskedastic error components, conditional on a set of covariates. For a truncated-normal distribution, `frontier` can also fit a conditional mean model in which the mean is modeled as a linear function of a set of covariates.

## ▷ Example 1: The half-normal and the exponential models

For our first example, we demonstrate the half-normal and exponential models by reproducing a study found in Greene (2003, 505), which uses data originally published in Zellner and Revankar (1969). In this study of the transportation-equipment manufacturing industry, observations on value added, capital, and labor are used to estimate a Cobb–Douglas production function. The variable `lnv` is the log-transformed value added, `lnk` is the log-transformed capital, and `lnl` is the log-transformed labor. OLS estimates are compared with those from stochastic frontier models using both the half-normal and exponential distribution for the inefficiency term.

```
. use http://www.stata-press.com/data/r14/greene9
```

```
. regress lnv lnk ln1
```

Source	SS	df	MS	Number of obs	=	25
Model	44.1727741	2	22.086387	F(2, 22)	=	397.54
Residual	1.22225984	22	.055557265	Prob > F	=	0.0000
				R-squared	=	0.9731
				Adj R-squared	=	0.9706
Total	45.3950339	24	1.89145975	Root MSE	=	.23571

  

lnv	Coef.	Std. Err.	t	P> t	[95% Conf. Interval]
lnk	.2454281	.1068574	2.30	0.032	.0238193 .4670368
ln1	.805183	.1263336	6.37	0.000	.5431831 1.067183
_cons	1.844416	.2335928	7.90	0.000	1.359974 2.328858

```
. frontier lnv lnk ln1
```

```
Iteration 0: log likelihood = 2.3357572
Iteration 1: log likelihood = 2.4673009
Iteration 2: log likelihood = 2.4695125
Iteration 3: log likelihood = 2.4695222
Iteration 4: log likelihood = 2.4695222
```

```
Stoc. frontier normal/half-normal model
```

```
Number of obs = 25
```

```
Wald chi2(2) = 743.71
```

```
Log likelihood = 2.4695222
```

```
Prob > chi2 = 0.0000
```

lnv	Coef.	Std. Err.	z	P> z	[95% Conf. Interval]
lnk	.2585478	.098764	2.62	0.009	.0649738 .4521218
ln1	.7802451	.1199399	6.51	0.000	.5451672 1.015323
_cons	2.081135	.281641	7.39	0.000	1.529128 2.633141
/lnsig2v	-3.48401	.6195353	-5.62	0.000	-4.698277 -2.269743
/lnsig2u	-3.014599	1.11694	-2.70	0.007	-5.203761 -.8254368
sigma_v	.1751688	.0542616			.0954514 .3214633
sigma_u	.2215073	.1237052			.074134 .6618486
sigma2	.0797496	.0426989			-.0039388 .163438
lambda	1.264536	.1678684			.9355204 1.593552

```
LR test of sigma_u=0: chibar2(01) = 0.43
```

```
Prob >= chibar2 = 0.256
```

```
. predict double u_h, u
```

```
. frontier lnv lnk ln1, distribution(exponential)
```

```
Iteration 0: log likelihood = 2.7270659
Iteration 1: log likelihood = 2.8551532
Iteration 2: log likelihood = 2.8604815
Iteration 3: log likelihood = 2.8604897
Iteration 4: log likelihood = 2.8604897
```

```
Stoc. frontier normal/exponential model      Number of obs      =          25
                                                Wald chi2(2)       =         845.68
Log likelihood = 2.8604897                    Prob > chi2        =          0.0000
```

lnv	Coef.	Std. Err.	z	P> z	[95% Conf. Interval]	
lnk	.2624859	.0919988	2.85	0.004	.0821717	.4428002
ln1	.7703795	.1109569	6.94	0.000	.5529079	.9878511
_cons	2.069242	.2356159	8.78	0.000	1.607444	2.531041
/lnsig2v	-3.527598	.4486176	-7.86	0.000	-4.406873	-2.648324
/lnsig2u	-4.002457	.9274575	-4.32	0.000	-5.820241	-2.184674
sigma_v	.1713925	.0384448			.1104231	.2660258
sigma_u	.1351691	.0626818			.0544692	.3354317
sigma2	.0476461	.0157921			.016694	.0785981
lambda	.7886525	.087684			.616795	.9605101

```
LR test of sigma_u=0: chibar2(01) = 1.21          Prob >= chibar2 = 0.135
```

```
. predict double u_e, u
```

```
. list state u_h u_e
```

	state	u_h	u_e
1.	Alabama	.2011338	.14592865
2.	California	.14480966	.0972165
3.	Connecticut	.1903485	.13478797
4.	Florida	.51753139	.5903303
5.	Georgia	.10397912	.07140994
6.	Illinois	.12126696	.0830415
7.	Indiana	.21128212	.15450664
8.	Iowa	.24933153	.20073081
9.	Kansas	.10099517	.06857629
10.	Kentucky	.05626919	.04152443
11.	Louisiana	.20332731	.15066405
12.	Maine	.22263164	.17245793
13.	Maryland	.13534062	.09245501
14.	Massachusetts	.15636999	.10932923
15.	Michigan	.15809566	.10756915
16.	Missouri	.10288047	.0704146
17.	NewJersey	.09584337	.06587986
18.	NewYork	.27787793	.22249416
19.	Ohio	.22914231	.16981857
20.	Pennsylvania	.1500667	.10302905
21.	Texas	.20297875	.14552218
22.	Virginia	.14000132	.09676078
23.	Washington	.11047581	.07533251
24.	WestVirginia	.15561392	.11236153
25.	Wisconsin	.14067066	.0970861

The parameter estimates and the estimates of the inefficiency terms closely match those published in [Greene \(2003, 505\)](#), but the standard errors of the parameter estimates are estimated differently (see the technical note below).

The output from `frontier` includes estimates of the standard deviations of the two error components,  $\sigma_v$  and  $\sigma_u$ , which are labeled `sigma_v` and `sigma_u`, respectively. In the log likelihood, they are parameterized as  $\ln\sigma_v^2$  and  $\ln\sigma_u^2$ , and these estimates are labeled `/lnsig2v` and `/lnsig2u` in the output. `frontier` also reports two other useful parameterizations. The estimate of the total error variance,  $\sigma_S^2 = \sigma_v^2 + \sigma_u^2$ , is labeled `sigma2`, and the estimate of the ratio of the standard deviation of the inefficiency component to the standard deviation of the idiosyncratic component,  $\lambda = \sigma_u/\sigma_v$ , is labeled `lambda`.

At the bottom of the output, `frontier` reports the results of a test that there is no technical inefficiency component in the model. This is a test of the null hypothesis  $H_0 : \sigma_u^2 = 0$  against the alternative hypotheses  $H_1 : \sigma_u^2 > 0$ . If the null hypothesis is true, the stochastic frontier model reduces to an OLS model with normal errors. However, because the test lies on the boundary of the parameter space of  $\sigma_u^2$ , the standard likelihood-ratio test is not valid, and a one-sided generalized likelihood-ratio test must be constructed; see [Gutierrez, Carter, and Drukker \(2001\)](#). For this example, the output shows LR = 0.43 with a  $p$ -value of 0.256 for the half-normal model and LR = 1.21 with a  $p$ -value of 0.135 for the exponential model. There are several possible reasons for the failure to reject the null hypothesis, but the fact that the test is based on an asymptotic distribution and the sample size was 25 is certainly a leading candidate among those possibilities.

◀

## □ Technical note

`frontier` maximizes the log-likelihood function of a stochastic frontier model by using the Newton–Raphson method, and the estimated variance–covariance matrix is calculated as the inverse of the negative Hessian (matrix of second partial derivatives); see [\[R\] ml](#). When comparing the results with those published using other software, be aware of the difference in the optimization methods, which may result in different, yet asymptotically equivalent, variance estimates.

□

## ▷ Example 2: Models with heteroskedasticity

Often the error terms may not have constant variance. `frontier` allows you to model heteroskedasticity in either error term as a linear function of a set of covariates. The variance of either the technical inefficiency or the idiosyncratic component may be modeled as

$$\sigma_i^2 = \exp(\mathbf{w}_i\boldsymbol{\delta})$$

The default constant included in  $\mathbf{w}_i$  may be suppressed by appending a `noconstant` option to the list of covariates. Also, you can simultaneously specify covariates for both  $\sigma_{u_i}$  and  $\sigma_{v_i}$ .

In this example, we use a sample of 756 observations of fictional firms producing a manufactured good by using capital and labor. The firms are hypothesized to use a constant returns-to-scale technology, but the sizes of the firms differ. Believing that this size variation will introduce heteroskedasticity into the idiosyncratic error term, we estimate the parameters of a Cobb–Douglas production function. To do this, we use a conditional heteroskedastic half-normal model, with the size of the firm as an explanatory variable in the variance function for the idiosyncratic error. We also perform a test of the hypothesis that the firms use a constant returns-to-scale technology.



```

. use http://www.stata-press.com/data/r14/frontier1, clear
. frontier lnoutput lnlabor lncapital, vhet(size)
Iteration 0:  log likelihood = -1508.3692
Iteration 1:  log likelihood = -1501.583
Iteration 2:  log likelihood = -1500.3942
Iteration 3:  log likelihood = -1500.3794
Iteration 4:  log likelihood = -1500.3794
Stoc. frontier normal/half-normal model      Number of obs   =       756
                                                Wald chi2(2)    =       9.68
Log likelihood = -1500.3794                  Prob > chi2     =       0.0079

```

lnoutput	Coef.	Std. Err.	z	P> z	[95% Conf. Interval]	
<b>lnoutput</b>						
lnlabor	.7090933	.2349374	3.02	0.003	.2486244	1.169562
lncapital	.3931345	.5422173	0.73	0.468	-.6695919	1.455861
_cons	1.252199	3.14656	0.40	0.691	-4.914946	7.419344
<b>lnsig2v</b>						
size	-.0016951	.0004748	-3.57	0.000	-.0026256	-.0007645
_cons	3.156091	.9265826	3.41	0.001	1.340023	4.97216
<b>lnsig2u</b>						
_cons	1.947487	.1017653	19.14	0.000	1.748031	2.146943
<b>sigma_u</b>						
sigma_u	2.647838	.134729			2.396514	2.925518

```

. test _b[lnlabor] + _b[lncapital] = 1
( 1)  [lnoutput]lnlabor + [lnoutput]lncapital = 1
      chi2( 1) =    0.03
      Prob > chi2 =   0.8622

```

The output above indicates that the variance of the idiosyncratic error term is a function of firm size. Also, we failed to reject the hypothesis that the firms use a constant returns-to-scale technology. ◀

## □ Technical note

In small samples, the conditional heteroskedastic estimators will lack precision for the variance parameters and may fail to converge altogether. □

## ▷ Example 3: The truncated-normal model

Let's turn our attention to the truncated-normal model. Once again, we will use fictional data. For this example, we have 1,231 observations on the quantity of output, the total cost of production for each firm, the prices that each firm paid for labor and capital services, and a categorical variable measuring the quality of each firm's management. After taking the natural logarithm of the costs (`lncost`), prices (`lnp_k` and `lnp_l`), and output (`lnout`), we fit a stochastic cost frontier model and specify the distribution for the inefficiency term to be truncated normal.

```

. use http://www.stata-press.com/data/r14/frontier2
. frontier lncost lnp_k lnp_l lnout, distribution(tnormal) cost

Iteration 0:  log likelihood = -2386.9523
Iteration 1:  log likelihood = -2386.5146
Iteration 2:  log likelihood = -2386.2704
Iteration 3:  log likelihood = -2386.2504
Iteration 4:  log likelihood = -2386.2493
Iteration 5:  log likelihood = -2386.2493

Stoc. frontier normal/truncated-normal model      Number of obs      =      1,231
                                                    Wald chi2(3)       =      8.82
Log likelihood = -2386.2493                       Prob > chi2        =      0.0318

```

lncost	Coef.	Std. Err.	z	P> z	[95% Conf. Interval]
lnp_k	.3410717	.2363861	1.44	0.149	-.1222366 .80438
lnp_l	.6608628	.4951499	1.33	0.182	-.3096131 1.631339
lnout	.7528653	.3468968	2.17	0.030	.0729601 1.432771
_cons	2.602609	1.083004	2.40	0.016	.4799595 4.725259
/mu	1.095705	.881517	1.24	0.214	-.632037 2.823446
/lnsigma2	1.5534	.1873464	8.29	0.000	1.186208 1.920592
/ilgtgamma	1.257862	.2589522	4.86	0.000	.7503255 1.765399
sigma2	4.727518	.8856833			3.274641 6.825001
gamma	.7786579	.0446303			.6792496 .8538846
sigma_u2	3.681119	.7503408			2.210478 5.15176
sigma_v2	1.046399	.2660035			.5250413 1.567756

H0: No inefficiency component:                      z = 5.595                      Prob>=z = 0.000

In addition to the coefficients, the output reports estimates for several parameters. `sigma_v2` is the estimate of  $\sigma_v^2$ . `sigma_u2` is the estimate of  $\sigma_u^2$ . `gamma` is the estimate of  $\gamma = \sigma_u^2 / \sigma_S^2$ . `sigma2` is the estimate of  $\sigma_S^2 = \sigma_v^2 + \sigma_u^2$ . Because  $\gamma$  must be between 0 and 1, the optimization is parameterized in terms of the logit of  $\gamma$ , and this estimate is reported as `ilgtgamma`. Because  $\sigma_S^2$  must be positive, the optimization is parameterized in terms of  $\ln(\sigma_S^2)$ , whose estimate is reported as `lnsigma2`. Finally, `mu` is the estimate of  $\mu$ , the mean of the truncated-normal distribution.

In the output above, the generalized log-likelihood test for the presence of the inefficiency term has been replaced with a test based on the third moment of the OLS residuals. When  $\mu = 0$  and  $\sigma_u = 0$ , the truncated-normal model reduces to a linear regression model with normally distributed errors. However, the distribution of the test statistic under the null hypothesis is not well established, because it becomes impossible to evaluate the log likelihood as  $\sigma_u$  approaches zero, prohibiting the use of the likelihood-ratio test.

However, [Coelli \(1995\)](#) noted that the presence of an inefficiency term would negatively skew the residuals from an OLS regression. By identifying negative skewness in the residuals with the presence of an inefficiency term, Coelli derived a one-sided test for the presence of the inefficiency term. The results of this test are given at the bottom of the output. For this example, the null hypothesis of no inefficiency component is rejected.

In the example below, we fit a truncated model and detect a statistically significant inefficiency term in the model. We might question whether the inefficiency term is identically distributed over all firms or whether there might be heterogeneity across firms. `frontier` provides an extension to the truncated normal model by allowing the mean of the inefficiency term to be modeled as a linear function of a set of covariates. In our dataset, we have a categorical variable that measures the quality of a firm's management. We refit the model, including the `cm()` option, specifying a set of

binary indicator variables representing the different categories of the quality-measurement variable as covariates.

```
. frontier lncost lnp_k lnp_l lnout, distribution(tnormal) cm(i.quality) cost
Iteration 0:  log likelihood = -2386.9523
Iteration 1:  log likelihood = -2384.936
Iteration 2:  log likelihood = -2382.3942
Iteration 3:  log likelihood = -2382.324
Iteration 4:  log likelihood = -2382.3233
Iteration 5:  log likelihood = -2382.3233
Stoc. frontier normal/truncated-normal model   Number of obs   =       1,231
                                                Wald chi2(3)    =         9.31
                                                Prob > chi2     =       0.0254
Log likelihood = -2382.3233
```

	lncost	Coef.	Std. Err.	z	P> z	[95% Conf. Interval]	
<b>lncost</b>							
	lnp_k	.3611204	.2359749	1.53	0.126	-.1013819	.8236227
	lnp_l	.680446	.4934935	1.38	0.168	-.2867835	1.647675
	lnout	.7605533	.3466102	2.19	0.028	.0812098	1.439897
	_cons	2.550769	1.078911	2.36	0.018	.4361417	4.665396
<b>mu</b>							
	quality						
	2	.5056067	.3382907	1.49	0.135	-.1574309	1.168644
	3	.783223	.376807	2.08	0.038	.0446947	1.521751
	4	.5577511	.3355061	1.66	0.096	-.0998288	1.215331
	5	.6792882	.3428073	1.98	0.048	.0073981	1.351178
	_cons	.6014025	.990167	0.61	0.544	-1.339289	2.542094
	/lnsigma2	1.541784	.1790926	8.61	0.000	1.190769	1.892799
	/ilgtgamma	1.242302	.2588968	4.80	0.000	.734874	1.749731
	sigma2	4.67292	.8368852			3.289611	6.637923
	gamma	.7759645	.0450075			.6758739	.8519189
	sigma_u2	3.62602	.7139576			2.226689	5.025351
	sigma_v2	1.0469	.2583469			.5405491	1.553251

The conditional mean model was developed in the context of panel-data estimators, and we can apply frontier's conditional mean model to panel data.

## Stored results

frontier stores the following in `e()`:

### Scalars

<code>e(N)</code>	number of observations
<code>e(df_m)</code>	model degrees of freedom
<code>e(k)</code>	number of parameters
<code>e(k_eq)</code>	number of equations in <code>e(b)</code>
<code>e(k_eq_model)</code>	number of equations in overall model test
<code>e(k_dv)</code>	number of dependent variables
<code>e(chi2)</code>	$\chi^2$
<code>e(ll)</code>	log likelihood
<code>e(ll_c)</code>	log likelihood for $H_0: \sigma_u=0$
<code>e(z)</code>	test for negative skewness of OLS residuals
<code>e(sigma_u)</code>	standard deviation of technical inefficiency
<code>e(sigma_v)</code>	standard deviation of $v_i$
<code>e(p)</code>	significance
<code>e(chi2_c)</code>	LR test statistic
<code>e(p_z)</code>	$p$ -value for $z$
<code>e(rank)</code>	rank of <code>e(V)</code>
<code>e(ic)</code>	number of iterations
<code>e(rc)</code>	return code
<code>e(converged)</code>	1 if converged, 0 otherwise

### Macros

<code>e(cmd)</code>	<code>frontier</code>
<code>e(cmdline)</code>	command as typed
<code>e(depvar)</code>	name of dependent variable
<code>e(function)</code>	production or cost
<code>e(wtype)</code>	weight type
<code>e(wexp)</code>	weight expression
<code>e(title)</code>	title in estimation output
<code>e(chi2type)</code>	Wald; type of model $\chi^2$ test
<code>e(dist)</code>	distribution assumption for $u_i$
<code>e(het)</code>	heteroskedastic components
<code>e(u_hetvar)</code>	<code>varlist</code> in <code>uhet()</code>
<code>e(v_hetvar)</code>	<code>varlist</code> in <code>vhet()</code>
<code>e(vce)</code>	<code>vcetype</code> specified in <code>vce()</code>
<code>e(vcetype)</code>	title used to label Std. Err.
<code>e(opt)</code>	type of optimization
<code>e(which)</code>	max or min; whether optimizer is to perform maximization or minimization
<code>e(ml_method)</code>	type of ml method
<code>e(user)</code>	name of likelihood-evaluator program
<code>e(technique)</code>	maximization technique
<code>e(properties)</code>	<code>b V</code>
<code>e(predict)</code>	program used to implement <code>predict</code>
<code>e(asbalanced)</code>	factor variables <code>fvset</code> as <code>asbalanced</code>
<code>e(asobserved)</code>	factor variables <code>fvset</code> as <code>asobserved</code>

### Matrices

<code>e(b)</code>	coefficient vector
<code>e(Cns)</code>	constraints matrix
<code>e(ilog)</code>	iteration log (up to 20 iterations)
<code>e(gradient)</code>	gradient vector
<code>e(V)</code>	variance-covariance matrix of the estimators
<code>e(V_modelbased)</code>	model-based variance

### Functions

<code>e(sample)</code>	marks estimation sample
------------------------	-------------------------

## Methods and formulas

Consider an equation of the form

$$y_i = \mathbf{x}_i\boldsymbol{\beta} + v_i - su_i$$

where  $y_i$  is the dependent variable,  $\mathbf{x}_i$  is a  $1 \times k$  vector of observations on the independent variables included as indent covariates,  $\boldsymbol{\beta}$  is a  $k \times 1$  vector of coefficients, and

$$s = \begin{cases} 1, & \text{for production functions} \\ -1, & \text{for cost functions} \end{cases}$$

The log-likelihood functions are as follows.

Normal/half-normal model:

$$\ln L = \sum_{i=1}^N \left\{ \frac{1}{2} \ln \left( \frac{2}{\pi} \right) - \ln \sigma_S + \ln \Phi \left( -\frac{s\epsilon_i \lambda}{\sigma_S} \right) - \frac{\epsilon_i^2}{2\sigma_S^2} \right\}$$

Normal/exponential model:

$$\ln L = \sum_{i=1}^N \left\{ -\ln \sigma_u + \frac{\sigma_v^2}{2\sigma_u^2} + \ln \Phi \left( \frac{-s\epsilon_i - \frac{\sigma_v^2}{\sigma_u}}{\sigma_v} \right) + \frac{s\epsilon_i}{\sigma_u} \right\}$$

Normal/truncated-normal model:

$$\begin{aligned} \ln L = \sum_{i=1}^N \left\{ -\frac{1}{2} \ln(2\pi) - \ln \sigma_S - \ln \Phi \left( \frac{\mu}{\sigma_S \sqrt{\gamma}} \right) \right. \\ \left. + \ln \Phi \left[ \frac{(1-\gamma)\mu - s\gamma\epsilon_i}{\{\sigma_S^2\gamma(1-\gamma)\}^{1/2}} \right] - \frac{1}{2} \left( \frac{\epsilon_i + s\mu}{\sigma_S} \right)^2 \right\} \end{aligned}$$

where  $\sigma_S = (\sigma_u^2 + \sigma_v^2)^{1/2}$ ,  $\lambda = \sigma_u/\sigma_v$ ,  $\gamma = \sigma_u^2/\sigma_S^2$ ,  $\epsilon_i = y_i - \mathbf{x}_i\boldsymbol{\beta}$ , and  $\Phi(\cdot)$  is the cumulative distribution function of the standard normal distribution.

To obtain estimation for  $u_i$ , you can use either the mean or the mode of the conditional distribution  $f(u|\epsilon)$ .

$$E(u_i | \epsilon_i) = \mu_{*i} + \sigma_* \left\{ \frac{\phi(-\mu_{*i}/\sigma_*)}{\Phi(\mu_{*i}/\sigma_*)} \right\}$$

$$M(u_i | \epsilon_i) = \begin{cases} \mu_{*i} & \text{if } \mu_{*i} \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Then the technical efficiency ( $s = 1$ ) or cost efficiency ( $s = -1$ ) will be estimated by

$$\begin{aligned} E_i &= E \{ \exp(-su_i) | \epsilon_i \} \\ &= \left\{ \frac{1 - \Phi(s\sigma_* - \mu_{*i}/\sigma_*)}{1 - \Phi(-\mu_{*i}/\sigma_*)} \right\} \exp \left( -s\mu_{*i} + \frac{1}{2}\sigma_*^2 \right) \end{aligned}$$

where  $\mu_{*i}$  and  $\sigma_*$  are defined for the normal/half-normal model as

$$\begin{aligned}\mu_{*i} &= -s\epsilon_i\sigma_u^2/\sigma_S^2 \\ \sigma_* &= \sigma_u\sigma_v/\sigma_S\end{aligned}$$

for the normal/exponential model as

$$\begin{aligned}\mu_{*i} &= -s\epsilon_i - \sigma_v^2/\sigma_u \\ \sigma_* &= \sigma_v\end{aligned}$$

and for the normal/truncated-normal model as

$$\begin{aligned}\mu_{*i} &= \frac{-s\epsilon_i\sigma_u^2 + \mu\sigma_v^2}{\sigma_S^2} \\ \sigma_* &= \sigma_u\sigma_v/\sigma_S\end{aligned}$$

In the half-normal and exponential models, when heteroskedasticity is assumed, the standard deviations,  $\sigma_u$  or  $\sigma_v$ , will be replaced in the above equations by

$$\sigma_i^2 = \exp(\mathbf{w}_i\boldsymbol{\delta})$$

where  $\mathbf{w}$  is the vector of explanatory variables in the variance function.

In the conditional mean model, the mean parameter of the truncated normal distribution,  $\mu$ , is modeled as a linear combination of the set of covariates,  $\mathbf{w}$ .

$$\mu = \mathbf{w}_i\boldsymbol{\delta}$$

Therefore, the log-likelihood function can be rewritten as

$$\begin{aligned}\ln L &= \sum_{i=1}^N \left[ -\frac{1}{2} \ln(2\pi) - \ln\sigma_S - \ln\Phi\left(\frac{\mathbf{w}_i\boldsymbol{\delta}}{\sqrt{\sigma_S^2\gamma}}\right) \right. \\ &\quad \left. + \ln\Phi\left\{\frac{(1-\gamma)\mathbf{w}_i\boldsymbol{\delta} - s\gamma\epsilon_i}{\sqrt{\sigma_S^2\gamma(1-\gamma)}}\right\} - \frac{1}{2}\left(\frac{\epsilon_i + s\mathbf{w}_i\boldsymbol{\delta}}{\sigma_S}\right)^2 \right]\end{aligned}$$

The  $z$  test reported in the output of the truncated-normal model is a third-moment test developed by Coelli (1995) as an extension of a test previously developed by Pagan and Hall (1983). Coelli shows that under the null of normally distributed errors, the statistic

$$z = \frac{m_3}{\left(\frac{6m_2^3}{N}\right)^{1/2}}$$

has a standard normal distribution, where  $m_3$  is the third moment from the OLS regression. Because the residuals are either negatively skewed (production function) or positively skewed (cost function), a one-sided  $p$ -value is used.

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## Also see

- [R] [frontier postestimation](#) — Postestimation tools for frontier
- [R] [regress](#) — Linear regression
- [XT] [xtfrontier](#) — Stochastic frontier models for panel data
- [U] [20 Estimation and postestimation commands](#)