Title

lusolve() — Solve AX=B for X using LU decomposition

Description

`lusolve(`*A*`,` *B*`)` solves *AX=B* and returns *X*. `lusolve()` returns a matrix of missing values if *A* is singular.

`lusolve(`*A*`,` *B*`,` *tol*`)` does the same thing but allows you to specify the tolerance for declaring that *A* is singular; see *Tolerance* under *Remarks and examples* below.

`_lusolve(`*A*`,` *B*`)` and `_lusolve(`*A*`,` *B*`,` *tol*`)` do the same thing except that, rather than returning the solution *X*, they overwrite *B* with the solution and, in the process of making the calculation, they destroy the contents of *A*.

`_lusolve_la(`*A*`,` *B*`)` and `_lusolve_la(`*A*`,` *B*`,` *tol*`)` are the interfaces to the [M-1] **LAPACK** routines that do the work. They solve *AX=B* for *X*, returning the solution in *B* and, in the process, using as workspace (overwriting) *A*. The routines return 1 if *A* was singular and 0 otherwise. If *A* was singular, *B* is overwritten with a matrix of missing values.

Syntax

numeric matrix	`lusolve(`*numeric matrix A*`,` *numeric matrix B*`)`
numeric matrix	`lusolve(`*numeric matrix A*`,` *numeric matrix B*`,` *real scalar tol*`)`
void	`_lusolve(`*numeric matrix A*`,` *numeric matrix B*`)`
void	`_lusolve(`*numeric matrix A*`,` *numeric matrix B*`,` *real scalar tol*`)`
real scalar	`_lusolve_la(`*numeric matrix A*`,` *numeric matrix B*`)`
real scalar	`_lusolve_la(`*numeric matrix A*`,` *numeric matrix B*`,` *real scalar tol*`)`

Remarks and examples

The above functions solve *AX=B* via LU decomposition and are accurate. An alternative is `qrsolve()` (see [M-5] **qrsolve()**), which uses QR decomposition. The difference between the two solutions is not, practically speaking, accuracy. When *A* is of full rank, both routines return equivalent results, and the LU approach is quicker, using approximately $O(2/3n^3)$ operations rather than $O(4/3n^3)$, where *A* is $n \times n$.

The difference arises when *A* is singular. Then the LU-based routines documented here return missing values. The QR-based routines documented in [M-5] **qrsolve()** return a generalized (least squares) solution.

For more information on LU and QR decomposition, see [M-5] **lud()** and see [M-5] **qrd()**.

Remarks are presented under the following headings:

Derivation

We wish to solve for X

$$AX = B \tag{1}$$

Perform LU decomposition on A so that we have $A = PLU$. Then (1) can be written as

$$PLUX = B$$

or, premultiplying by P' and remembering that $P'P = I$,

$$LUX = P'B \tag{2}$$

Define

$$Z = UX \tag{3}$$

Then (2) can be rewritten as

$$LZ = P'B \tag{4}$$

It is easy to solve (4) for Z because L is a lower-triangular matrix. Once Z is known, it is easy to solve (3) for X because U is upper triangular.

Relationship to inversion

Another way to solve

$$AX = B$$

is to obtain A^{-1} and then calculate

$$X = A^{-1}B$$

It is, however, better to solve $AX = B$ directly because fewer numerical operations are required, and the result is therefore more accurate and obtained in less computer time.

Indeed, rather than thinking about how solving a system of equations can be implemented via inversion, it is more productive to think about how inversion can be implemented via solving a system of equations. Obtaining A^{-1} amounts to solving

$$AX = I$$

Thus `lusolve()` (or any other solve routine) can be used to obtain inverses. The inverse of A can be obtained by coding

```
: Ainv = lusolve(A, I(rows(A)))
```

In fact, we provide `luinv()` (see [M-5] **luinv()**) for obtaining inverses via LU decomposition, but `luinv()` amounts to making the above calculation, although a little memory is saved because the matrix I is never constructed.

Hence, everything said about `lusolve()` applies equally to `luinv()`.

Tolerance

The default tolerance used is

eta = (1e-13)*trace(abs(*U*))/*n*

where U is the upper-triangular matrix of the LU decomposition of A: $n \times n$. A is declared to be singular if any diagonal element of U is less than or equal to *eta*.

If you specify $tol > 0$, the value you specify is used to multiply *eta*. You may instead specify $tol \leq 0$, and then the negative of the value you specify is used in place of *eta*; see [M-1] **tolerance**.

So why not specify $tol = 0$? You do not want to do that because, as matrices become close to being singular, results can become inaccurate. Here is an example:

```
: rseed(12345)
: A = lowertriangle(runiform(4,4))
: A[3,3] = 1e-15
: trux = runiform(4,1)
: b = A*trux
: /* the above created an Ax=b problem, and we have placed the true
>    value of x in trux.  We now obtain the solution via lusolve()
>    and compare trux with the value obtained:
> */
: x = lusolve(A, b, 0)
: trux, x
```

1	.260768733	.260768733
2	.0267289389	.0267289389
3	.1079423963	.0989119749
4	.3666839808	.3863636364

← The discussed numerical instability can cause this output to vary a little across different computers

We would like to see the second column being nearly equal to the first—the estimated x being nearly equal to the true x—but there are substantial differences.

Even though the difference between x and trux is substantial, the difference between them is small in the prediction space:

```
: A*trux-b, A*x-b
```

	1	2
1	0	0
2	0	0
3	0	0
4	0	0

What made this problem so difficult was the line A[3,3] = 1e-15. Remove that and you would find that the maximum absolute difference between x and trux would be 5.55112e−15.

The degree to which the residuals A*x-b are a reliable measure of the accuracy of x depends on the condition number of the matrix, which can be obtained by [M-5] **cond()**, which for A, is 4.81288e+15. If the matrix is well conditioned, small residuals imply an accurate solution for x. If the matrix is ill conditioned, small residuals are not a reliable indicator of accuracy.

Another way to check the accuracy of x is to set *tol* = 0 and to see how well x could be obtained were x = x:

```
: x  = lusolve(A, b, 0)
: x2 = lusolve(A, A*x, 0)
```

If x and x2 are virtually the same, then you can safely assume that x is the result of a numerically accurate calculation. You might compare x and x2 with mreldif(x2,x); see [M-5] **reldif( )**. In our example, mreldif(x2,x) is .03, a large difference.

If A is ill conditioned, then small changes in A or B can lead to radical differences in the solution for X.

Conformability

lusolve(*A*, *B*, *tol*):

input:

A:	$n \times n$	
B:	$n \times k$	
tol:	1×1	(optional)

output:

result:	$n \times k$	

_lusolve(*A*, *B*, *tol*):

input:

A:	$n \times n$	
B:	$n \times k$	
tol:	1×1	(optional)

output:

A:	0×0
B:	$n \times k$

_lusolve_la(*A*, *B*, *tol*):

input:

A:	$n \times n$	
B:	$n \times k$	
tol:	1×1	(optional)

output:

A:	0×0
B:	$n \times k$
result:	1×1

Diagnostics

lusolve(*A*, *B*, ...), _lusolve(*A*, *B*, ...), and _lusolve_la(*A*, *B*, ...) return a result containing missing if A or B contain missing values. The functions return a result containing all missing values if A is singular.

_lusolve(*A*, *B*, ...) and _lusolve_la(*A*, *B*, ...) abort with error if A or B is a view.

_lusolve_la(*A*, *B*, ...) should not be used directly; use _lusolve().

Also see

[M-5] **luinv( )** — Square matrix inversion

[M-5] **lud( )** — LU decomposition

[M-5] **solvelower( )** — Solve AX=B for X, A triangular

[M-5] **cholsolve( )** — Solve AX=B for X using Cholesky decomposition

[M-5] **qrsolve( )** — Solve AX=B for X using QR decomposition

[M-5] **svsolve( )** — Solve AX=B for X using singular value decomposition

[M-4] **matrix** — Matrix functions

[M-4] **solvers** — Functions to solve AX=B and to obtain A inverse