Title

geigensystem( ) — Generalized eigenvectors and eigenvalues

Description

`geigensystem(`*A*`,` *B*`,` *X*`,` *w*`,` *b*`)` computes generalized eigenvectors of two general, real or complex, square matrices, *A* and *B*, along with their corresponding generalized eigenvalues.

- *A* and *B* are two general, real or complex, square matrices with the same dimensions.
- *X* contains generalized eigenvectors.
- *w* contains numerators of generalized eigenvalues.
- *b* contains denominators of generalized eigenvalues.

`leftgeigensystem(`*A*`,` *B*`,` *X*`,` *w*`,` *b*`)` mirrors `geigensystem()`, the difference being that `left-geigensystem()` computes left, generalized eigenvectors.

`geigensystemselectr(`*A*`,` *B*`,` *range*`,` *X*`,` *w*`,` *b*`)` computes selected generalized eigenvectors of two general, real or complex, square matrices, *A* and *B*, along with their corresponding generalized eigenvalues. Only the generalized eigenvectors corresponding to selected generalized eigenvalues are computed. Generalized eigenvalues that lie in a range are selected. The selected generalized eigenvectors are returned in *X*, and their corresponding generalized eigenvalues are returned in (*w*, *b*).

> *range* is a vector of length 2. All finite, generalized eigenvalues with absolute value in the half-open interval (*range*`[1]`, *range*`[2]`] are selected.

`leftgeigensystemselectr(`*A*`,` *B*`,` *range*`,` *X*`,` *w*`,` *b*`)` mirrors `geigensystemselectr()`, the difference being that `leftgeigensystemr()` computes left, generalized eigenvectors.

`geigensystemselecti(`*A*`,` *B*`,` *index*`,` *X*`,` *w*`,` *b*`)` computes selected right, generalized eigenvectors of two general, real or complex, square matrices, *A* and *B*, along with their corresponding generalized eigenvalues. Only the generalized eigenvectors corresponding to selected generalized eigenvalues are computed. Generalized eigenvalues are selected by an index. The selected generalized eigenvectors are returned in *X*, and the selected generalized eigenvalues are returned in (*w*, *b*).

> The finite, generalized eigenvalues are sorted by their absolute values, in descending order, followed by the infinite, generalized eigenvalues. There is no particular order among infinite, generalized eigenvalues.
>
> *index* is a vector of length 2. The generalized eigenvalues in elements *index*`[1]` through *index*`[2]`, inclusive, are selected.

`leftgeigensystemselecti(`*A*`,` *B*`,` *index*`,` *X*`,` *w*`,` *b*`)` mirrors `geigensystemselecti()`, the difference being that `leftgeigensystemi()` computes left, generalized eigenvectors.

`geigensystemselectf(`*A*`,` *B*`,` *f*`,` *X*`,` *w*`,` *b*`)` computes selected generalized eigenvectors of two general, real or complex, square matrices *A* and *B* along with their corresponding generalized eigenvalues. Only the generalized eigenvectors corresponding to selected generalized eigenvalues are computed. Generalized eigenvalues are selected by a user-written function described below. The selected generalized eigenvectors are returned in *X*, and the selected generalized eigenvalues are returned in (*w*, *b*).

`leftgeigensystemselectf(`*A*`,` *B*`,` *f*`,` *X*`,` *w*`,` *b*`)` mirrors `geigensystemselectf()`, the difference being that `leftgeigensystemselectf()` computes selected left, generalized eigenvectors.

`_geigen_la()`, `_geigensystem_la()`, `_geigenselectr_la()`, `_geigenselecti_la()`, and `_geigenselectf_la()` are the interfaces into the LAPACK routines used to implement the above functions; see [M-1] **LAPACK**. Their direct use is not recommended.

Syntax

```
void              geigensystem(A, B, X, w, b)
void          leftgeigensystem(A, B, X, w, b)

void       geigensystemselectr(A, B, range, X, w, b)
void   leftgeigensystemselectr(A, B, range, X, w, b)

void       geigensystemselecti(A, B, index, X, w, b)
void   leftgeigensystemselecti(A, B, index, X, w, b)

void       geigensystemselectf(A, B, f, X, w, b)
void   leftgeigensystemselectf(A, B, f, X, w, b)
```

where inputs are

A:	*numeric matrix*	
B:	*numeric matrix*	
range:	*real vector*	(range of generalized eigenvalues to be selected)
index:	*real vector*	(indices of generalized eigenvalues to be selected)
f:	*pointer scalar*	(points to a function used to select generalized eigenvalues)

and outputs are

X: *numeric matrix* of generalized eigenvectors
w: *numeric vector* (numerators of generalized eigenvalues)
b: *numeric vector* (denominators of generalized eigenvalues)

The following routines are used in implementing the above routines:

void `_geigensystem_la(`*numeric matrix H*`,` *R*`,` *XL*`,` *XR*`,` *w*`,` *b*`,`
string scalar side`)`

void `_geigenselectr_la(`*numeric matrix H*`,` *R*`,` *XL*`,` *XR*`,` *w*`,` *b*`,`
range`,` *string scalar side*`)`

void `_geigenselecti_la(`*numeric matrix H*`,` *R*`,` *XL*`,` *XR*`,` *w*`,` *b*`,`
index`,` *string scalar side*`)`

void `_geigenselectf_la(`*numeric matrix H*`,` *R*`,` *XL*`,` *XR*`,` *w*`,` *b*`,`
pointer scalar f`,` *string scalar side*`)`

real scalar `_geigen_la(`*numeric matrix H*`,` *R*`,` *XL*`,` *XR*`,` *w*`,` *select*`,`
string scalar side`,` *string scalar howmany*`)`

Remarks and examples

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Remarks are presented under the following headings:

Generalized eigenvalues

A scalar, *l* (usually denoted by *lambda*), is said to be a generalized eigenvalue of a pair of $n \times n$ square, numeric matrices $(\mathbf{A}, \mathbf{B})$ if there is a nonzero column vector $\mathbf{x}$: $n \times 1$ (called the generalized eigenvector) such that

$$\mathbf{A}\mathbf{x} = l\mathbf{B}\mathbf{x} \tag{1}$$

(1) can also be written as

$$(\mathbf{A} - l\mathbf{B})\mathbf{x} = 0$$

A nontrivial solution to this system of *n* linear homogeneous equations exists if and only if

$$\det(\mathbf{A} - l\mathbf{B}) = 0 \tag{2}$$

In practice, the generalized eigenvalue problem for the matrix pair $(\mathbf{A}, \mathbf{B})$ is usually formulated as finding a pair of scalars (*w*, *b*) and a nonzero column vector $\mathbf{x}$ such that

$$w\mathbf{A}\mathbf{x} = b\mathbf{B}\mathbf{x}$$

The scalar w/b is a finite, generalized eigenvalue if *b* is not zero. The pair (*w*, *b*) represents an infinite, generalized eigenvalue if *b* is zero or numerically close to zero. This situation may arise if $\mathbf{B}$ is singular.

The Mata functions that compute generalized eigenvalues return them in two complex vectors, $\mathbf{w}$ and $\mathbf{b}$, of length *n*. If `b[`*i*`]=0`, the *i*th generalized eigenvalue is infinite; otherwise, the *i*th generalized eigenvalue is `w[`*i*`]/b[`*i*`]`.

Generalized eigenvectors

A column vector, $\mathbf{x}$, is a right, generalized eigenvector or simply a generalized eigenvector of a generalized eigenvalue (w, b) for a pair of matrices, $\mathbf{A}$ and $\mathbf{B}$, if

$$w\mathbf{A}\mathbf{x} = b\mathbf{B}\mathbf{x}$$

A row vector, $\mathbf{v}$, is a left, generalized eigenvector of a generalized eigenvalue (w, b) for a pair of matrices, $\mathbf{A}$ and $\mathbf{B}$, if

$$w\mathbf{v}\mathbf{A} = b\mathbf{v}\mathbf{B}$$

For instance, let's consider the linear system

$$dx/dt = \mathtt{A1} \times x + \mathtt{A2} \times u$$

$$dy/dt = \mathtt{A3} \times x + \mathtt{A4} \times u$$

where

```
: A1 = (-4, -3 \ 2, 1)
: A2 = (3 \ 1)
: A3 = (1, 2)
```

and

```
: A4 = 0
```

The finite solutions of zeros for the transfer function

$$g(s) = \mathtt{A3} \times (sI - \mathtt{A1})^{-1} \times \mathtt{A2} + \mathtt{A4} \tag{3}$$

of this linear time-invariant state-space model is given by the finite, generalized eigenvalues of `A` and `B` where

```
: A = (A1, A2 \ A3, A4)
```

and

```
: B = (1, 0, 0 \ 0, 1, 0 \ 0, 0, 0)
```

We obtain generalized eigenvectors in `X` and generalized eigenvalues in `w` and `b` by using

```
: geigensystem(A, B, X=., w=., b=.)
: X
        1    2             3
    +-------------------------+
  1 |  -1    0    2.9790e-16  |
  2 |  .5    0    9.9301e-17  |
  3 |  .1    1             1  |
    +-------------------------+

: w
                 1             2             3
    +-------------------------------------------+
  1 |  -1.97989899    3.16227766    2.23606798  |
    +-------------------------------------------+

: b
                 1             2             3
    +-------------------------------------------+
  1 |  .7071067812             0             0  |
    +-------------------------------------------+
```

The only finite, generalized eigenvalue of A and B is

```
: w[1,1]/b[1,1]
  -2.8
```

In this simple example, (3) can be explicitly written out as

$$g(s) = (5s + 14)/(s^2 + 3s + 2)$$

which clearly has the solution of zero at -2.8.

Criterion selection

We sometimes want to compute only those generalized eigenvectors whose corresponding generalized eigenvalues satisfy certain criterion. We can use `geigensystemselectf()` to solve these problems.

We must pass `geigensystemselectf()` a pointer to a function that implements our conditions. The function must accept two numeric scalar arguments so that it can receive the numerator `w` and the denominator `b` of a generalized eigenvalue, and it must return the real value 0 to indicate rejection and a nonzero real value to indicate selection.

In this example, we want to compute only finite, generalized eigenvalues for each of which `b` is not zero. After deciding that anything smaller than 1e–15 is zero, we define our function to be

```
: real scalar finiteonly(numeric scalar w, numeric scalar b)
> {
>         return((abs(b)>=1e-15))
> }
```

By using

```
: geigensystemselectf(A, B, &finiteonly(), X=., w=., b=.)
```

we get the only finite, generalized eigenvalue of A and B in (w, b) and its corresponding eigenvector in X:

```
: X
                  1
    +---------------+
  1 |  -.894427191  |
  2 |   .447213595  |
  3 |   .089442719  |
    +---------------+

: w
  -1.97989899

: b
  .7071067812

: w:/b
  -2.8
```

Range selection

We can use `geigensystemselectr()` to compute only those generalized eigenvectors whose generalized eigenvalues have absolute values that fall in a half-open interval.

For instance,

```
: A = (-132, -88, 84, 104 \ -158.4, -79.2, 76.8, 129.6 \
> 129.6, 81.6, -79.2, -100.8 \ 160, 84, -80, -132)
: B = (-60, -50, 40, 50 \ -69, -46.4, 38, 58.2 \ 58.8, 46, -37.6, -48 \
> 70, 50, -40, -60)
: range = (0.99, 2.1)
```

We obtain generalized eigenvectors in `X` and generalized eigenvalues in `w` and `b` by using

```
: geigensystemselectr(A, B, range, X=., w=., b=.)
: X
                1             2
    +-----------------------------+
  1 |  .089442719     .02236068   |
  2 |   .04472136    .067082039   |
  3 |   .04472136    .067082039   |
  4 |  .089442719     .02236068   |
    +-----------------------------+
: w
               1             2
    +-----------------------------+
  1 |  .02820603    .170176379    |
    +-----------------------------+
: b
                 1              2
    +-------------------------------+
  1 |  .0141030148    .1701763791   |
    +-------------------------------+
```

The generalized eigenvalues have absolute values in the half-open interval (0.99, 2.1].

```
: abs(w:/b)
        1   2
    +---------+
  1 |   2   1 |
    +---------+
```

Index selection

`geigensystemselect()` sorts the finite, generalized eigenvalues using their absolute values, in descending order, placing the infinite, generalized eigenvalues after the finite, generalized eigenvalues. There is no particular order among infinite, generalized eigenvalues.

If we want to compute only generalized eigenvalues whose ranks are *index*`[1]` through *index*`[2]` in the list of generalized eigenvalues obtained by `geigensystemselect()`, we can use `geigensystemseleci()`.

To compute the first two generalized eigenvalues and generalized eigenvectors in this example, we can specify

```
: index = (1, 2)
: geigensystemselecti(A, B, index, X=., w=., b=.)
```

The results are

```
: X
                  1              2
    +-----------------------------------+
  1 |     .02981424    -.059628479     |
  2 |     .04472136    -.059628479     |
  3 |    .089442719     -.02981424     |
  4 |     .01490712    -.119256959     |
    +-----------------------------------+

: w
                 1              2
    +-------------------------------+
  1 |  .012649111     .379473319    |
    +-------------------------------+

: b
                  1              2
    +----------------------------------+
  1 |  .0031622777    .1264911064      |
    +----------------------------------+

: w:/b
        1   2
    +---------+
  1 |   4   3 |
    +---------+
```

Conformability

geigensystem(A, B, X, w, b):

input:

A:	$n \times n$
B:	$n \times n$

output:

X:	$n \times n$
w:	$1 \times n$
b:	$1 \times n$

leftgeigensystem(A, B, X, w, b):

input:

A:	$n \times n$
B:	$n \times n$

output:

X:	$n \times n$
w:	$1 \times n$
b:	$1 \times n$

geigensystemselectr(A, B, *range*, X, w, b):

input:

A:	$n \times n$
B:	$n \times n$
range:	1×2 or 2×1

output:

X:	$n \times m$
w:	$1 \times m$
b:	$1 \times m$

`leftgeigensystemselectr(`*A*`,` *B*`,` *range*`,` *X*`,` *w*`,` *b*`):`

input:

A:	$n \times n$
B:	$n \times n$
range:	1×2 or 2×1

output:

X:	$m \times n$
w:	$1 \times m$
b:	$1 \times m$

`geigensystemselecti(`*A*`,` *B*`,` *index*`,` *X*`,` *w*`,` *b*`):`

input:

A:	$n \times n$
B:	$n \times n$
index:	1×2 or 2×1

output:

X:	$n \times m$
w:	$1 \times m$
b:	$1 \times m$

`leftgeigensystemselecti(`*A*`,` *B*`,` *index*`,` *X*`,` *w*`,` *b*`):`

input:

A:	$n \times n$
B:	$n \times n$
index:	1×2 or 2×1

output:

X:	$m \times n$
w:	$1 \times m$
b:	$1 \times m$

`geigensystemselectf(`*A*`,` *B*`,` *f*`,` *X*`,` *w*`,` *b*`):`

input:

A:	$n \times n$
B:	$n \times n$
f:	1×1

output:

X:	$n \times m$
w:	$1 \times m$
b:	$1 \times m$

`leftgeigensystemselectf(`*A*`,` *B*`,` *f*`,` *X*`,` *w*`,` *b*`):`

input:

A:	$n \times n$
B:	$n \times n$
f:	1×1

output:

X:	$m \times n$
w:	$1 \times m$
b:	$1 \times m$

Diagnostics

All functions return missing-value results if *A* or *B* has missing values.

References

Gould, W. W. 2011a. Understanding matrices intuitively, part 1. *The Stata Blog: Not Elsewhere Classified.* http://blog.stata.com/2011/03/03/understanding-matrices-intuitively-part-1/.

——. 2011b. Understanding matrices intuitively, part 2, eigenvalues and eigenvectors. *The Stata Blog: Not Elsewhere Classified.* http://blog.stata.com/2011/03/09/understanding-matrices-intuitively-part-2/.

Also see

[M-1] **LAPACK** — The LAPACK linear-algebra routines

[M-5] **ghessenbergd( )** — Generalized Hessenberg decomposition

[M-5] **gschurd( )** — Generalized Schur decomposition

[M-4] **matrix** — Matrix functions